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The AMERICAN MATHEMATICAL MONTHLY

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THE INTEGRABILITY OF CERTAIN FUNCTIONS AND RELATED SUMMABILITY METHODS*

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1. Introduction. This paper was especially written for the MONTHLY because it shows that a closer analysis of an old textbook example involves in a natural way a variety of subjects such as integrability, summability, and the asymptotic distribution of sequences. The most familiar example of a function $f(x)$ which is not Riemann integrable is due to Dirichlet ([2], p. 132): $f(x)$ is defined in the range $[0, 1]$ as being 0 if x is irrational and 1 if x is rational. In many textbooks (see *e.g.*, [1], p. 83) we find the function defined in $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/q & \text{if } x = p/q, \quad (p, q) = 1, \quad 0 = 0/1, \end{cases}$$

where it is proposed as an exercise to show that $f(x)$ is discontinuous at all rationals and continuous at all irrationals. For indeed, if ξ is irrational and $p/q \to \xi$, then $q \to \infty$ so that

$$\lim f(p/q) = \lim 1/q = 0 = f(\xi).$$

Its discontinuity at rational points is clear. Thus the set of discontinuity points of $f(x)$ is a set of measure zero; the function being bounded we conclude by a classical criterion that $f(x)$ is Riemann integrable in the range $[0, 1]$ (see [4], Th. 1, p. 208).

We are here concerned with the following generalization of these functions: Let

$$\{\gamma_n\}, \qquad n = 1, 2, \cdots, \tag{1.1}$$

be a given sequence of reals and let $f(x)$ be defined in $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \gamma_q & \text{if } x = p/q, \quad (p, q) = 1. \end{cases} \tag{1.2}$$

When is this function Riemann integrable? We have just seen that it is integrable if $\gamma_n = 1/n$, but not integrable if $\gamma_n = 1$.

Let us now assume that

$$\lim \gamma_n = 0 \tag{1.3}$$

and let us show that $f(x)$ is then R-integrable. Indeed, the simple argument mentioned above applies also now. If ξ is irrational and $p/q \to \xi$, then $q \to \infty$, hence by (1.3)

$$\lim f(p/q) = \lim \gamma_q = 0 = f(\xi).$$

* The present work was partly sponsored by the Air Research and Development Command, USAF, at the University of Pennsylvania.

Hence (1.3) implies the continuity of $f(x)$ at all irrationals. Since $f(x)$ is evidently bounded, we again conclude as above that our function is integrable. However, the converse is also true as stated in the following

THEOREM 1. *The function $f(x)$ defined by* (1.1) *and* (1.2) *is Riemann integrable in the range* $[0, 1]$ *if and only if* (1.3) *holds.*

The sufficiency of (1.3) has already been shown. To prove the necessity let us assume that (1.3) does *not* hold and let us show that $f(x)$ *is discontinuous at every irrational point* $x=\xi$.

To prove this let n be given and let us consider the $\phi(n)$ integers $r_1, r_2, \cdots, r_\phi$ in natural order, which do not exceed n and are prime to n. The $\phi(n)$ reduced fractions

$$0 < \frac{r_1}{n} < \frac{r_2}{n} < \cdots < \frac{r_\phi}{n} < 1 \tag{1.4}$$

divide the interval $[0, 1]$ into $\phi+1$ subintervals. Let us denote by g_n the length of the largest of these subintervals. We shall now take for granted, proving it later, the interesting property

$$\lim_{n\to\infty} g_n = 0. \tag{1.5}$$

The discontinuity of $f(x)$ at irrational points is now easily established. For if ξ is irrational, $0<\xi<1$, then (1.5) shows that the inequalities

$$\frac{r_\nu}{n} < \xi < \frac{r_{\nu+1}}{n}$$

define uniquely the integer $\nu=\nu(n)$, provided that n is sufficiently large. Again (1.5) implies that

$$\lim_{n\to\infty} \frac{r_\nu}{n} = \xi. \tag{1.6}$$

Since $\gamma_n \nrightarrow 0$ by assumption, we obtain

$$f(r_\nu/n) = \gamma_n \nrightarrow 0 = f(\xi)$$

showing, in view of (1.6), that $f(x)$ is discontinuous at $x=\xi$.

The property (1.5), which we have just used, is a corollary of a stronger result due to G. Pólya (see [8] and [9], Problem 188, p. 75) to the effect that the set of fractions (1.4) is asymptotically equidistributed in the range $[0, 1]$. This means that if $g(x)$ is any Riemann integrable function in $[0, 1]$ then

$$\lim_{n\to\infty} \frac{g(r_1/n) + g(r_2/n) + \cdots + g(r_\phi/n)}{\phi(n)} = \int_0^1 g(x)dx. \tag{1.7}$$

That this implies the weaker statement (1.5) is seen as follows: Let us assume that (1.5) were false; this means that there exists a $\delta>0$ such that $g_n>\delta$ for arbitrarily large values of $n=n_\nu$ ($\nu=1, 2, \cdots$). An obvious application of the Bolzano-Weierstrass principle leads to the existence of an interval J, a subinterval of $[0, 1]$, such that J contains none of the fractions (1.4) if $n=n_\nu$. However, this clearly contradicts the relation (1.7) if

$$g(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0 & \text{if } x \notin J. \end{cases}$$

This completes our proof of Theorem 1.

In the next section it is shown how the Riemann sums (2.4) of our function $f(x)$ lead to the regular summability method (2.5) which seems not to have been studied before. This summability method turns out to be peculiarly weak: If a sequence $\{\gamma_n\}$ is summable by it to the limit λ, then $\gamma_{n_\nu}\to\lambda$ over subsequences $\{n_\nu\}$ which can be readily described (Theorem 2 below). Known results on the arithmetical function $\phi(n)/n$ allow to establish contact with the notion of an asymptotic distribution function of a sequence of which equidistribution in $[0, 1]$ is a notable example. The case when a sequence $\{\gamma_n\}$ has an asymptotic distribution function having only one point of increase (with unit jump) is interpreted as a generalization of the notion of convergence called D-convergence. It is shown that sequences summable by the method (2.5) are also D-convergent (Theorem 3). An analytic criterion for D-convergence is established by elementary means (Theorem 4). A last brief section records an interesting remark by H. Rademacher exhibiting sequences $\{\gamma_n\}$ such that the function (1.2) is not only continuous but also differentiable at algebraic irrationalities.

2. The Riemann sums of $f(x)$ and a related summability method. A trivial generalization of Theorem 1 is as follows. Let the sequence (1.1) be given as well as a fixed real number λ. Let $f(x)$ be defined in $[0, 1]$ by

$$f(x) = \begin{cases} \lambda & \text{if } x \text{ is irrational} \\ \gamma_q & \text{if } x = p/q, \quad (p, q) = 1. \end{cases} \tag{2.1}$$

Then $f(x)$ *is R-integrable if and only if*

$$\lim \gamma_n = \lambda. \tag{2.2}$$

Indeed, either repeat previous arguments or apply old result to $f(x)-\lambda$.

Let us now assume that (2.2) holds. Being R-integrable, $f(x)$ is also summable in the sense of Lebesgue, while

$$(R)\int_0^1 f(x)dx = (L)\int_0^1 f(x)dx = (L)\int_0^1 \lambda dx = \lambda,$$

since $f(x)=\lambda$ almost everywhere. The Riemann integral of $f(x)$ being λ we may

conclude that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} f\left(\frac{\nu}{n}\right) = \lambda. \tag{2.3}$$

On the other hand, the Riemann sums

$$s_n = \frac{1}{n} \sum_{\nu=1}^{n} f\left(\frac{\nu}{n}\right), \qquad (n = 1, 2, \cdots) \tag{2.4}$$

are evidently finite linear combinations of the elements of the sequence $\{\gamma_n\}$. If we interpret the relations (2.4) as a transformation which transforms the sequence $\{\gamma_n\}$ into the sequence $\{s_n\}$, we may restate our result ($\gamma_n \to \lambda$ implies $s_n \to \lambda$) by saying that (2.4) is a *regular transformation* (*i.e.*, convergence and limit preserving; see [5], p. 43).

However, the matter just discussed belongs to the realm of algebraic analysis and may be easily settled directly without using integration theory. Let us express the sums (2.4) in terms of the γ_ν. For this purpose we classify the integers $\nu = 1, \cdots, n$ by the value of their g.c.d. with n. If d' is a divisor of n, then the ν with $(n, \nu) = d'$ are given by $\nu = d'l$, $n = d'd$, where l is such that $1 \leqq l \leqq d$, $(l, d) = 1$. By re-grouping the terms of the sum (2.4) according to this classification of the ν, we obtain, since $\nu/n = l/d$, that

$$\sum_{\nu=1}^{n} f\left(\frac{\nu}{n}\right) = \sum_{d|n} \sum_{\substack{(l,d)=1 \\ l\leq d}} f\left(\frac{l}{d}\right) = \sum_{d|n} \sum_{\substack{(l,d)=1 \\ l\leq d}} \gamma_d = \sum_{d|n} \phi(d)\gamma_d.$$

In the Dirichlet case when all the $\gamma_n = 1$, also $f(\nu/n) = 1$, and we obtain

$$n = \sum_{d|n} \phi(d),$$

which is a theorem of Gauss ([6], p. 30). The explicit expression of the Riemann sums (2.4) now becomes

$$s_n = \frac{\sum_{d|n} \phi(d)\gamma_d}{\sum_{d|n} \phi(d)} \qquad (n = 1, 2, \cdots). \tag{2.5}$$

The above conclusion to the effect that $\gamma_n \to \lambda$ implies $s_n \to \lambda$ now becomes again evident because the transformation (2.5) satisfies the conditions for a regular transformation given by the theorem of Toeplitz (see [5], Th. 2, p. 43). Indeed, notice that on writing (2.5) in the form

$$s_n = \sum_{\nu} a_{n\nu}\gamma_\nu,$$

we have $a_{n\nu} \geqq 0$, $\sum_\nu a_{n\nu} = 1$, $a_{n\nu} \to 0$ as $n \to \infty$ for each fixed ν.

Let us say for convenience that *the sequence* $\{\gamma_n\}$ *is* ϕ*-summable, or* ϕ*-con-*

vergent, to the ϕ-limit λ and write

$$\phi\text{-}\lim \gamma_n = \lambda, \tag{2.6}$$

meaning thereby that the transformed sequence $\{s_n\}$ has the property

$$\lim s_n = \lambda. \tag{2.7}$$

We know by Theorem 1 that

$$\lim \gamma_n = \lambda \tag{2.8}$$

is necessary and sufficient for the R-integrability of the function (2.1). Also (2.3), or (2.7), is clearly a *necessary* condition. However, it is conceivable that the relation (2.7) might also be sufficient for the R-integrability of $f(x)$. Such would be the case if and only if the ϕ-convergence of a sequence $\{\gamma_n\}$ would imply its ordinary convergence. Such, however, is not the case as will be shown in the next section.

3. Properties of the transformation inverse to (2.5). Let us invert the transformation (2.5) or

$$ns_n = \sum_{d|n} \phi(d)\gamma_d.$$

By Moebius' inversion formula (see *e.g.* [6], p. 87) we obtain

$$\phi(n)\gamma_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) ds_d, \tag{3.1}$$

or

$$\gamma_n = \frac{1}{\phi(n)} \sum_{d|n} \mu\left(\frac{n}{d}\right) ds_d \qquad (n = 1, 2, \cdots). \tag{3.2}$$

Also from (3.1), for all $\gamma_\nu = 1$ hence all $s_\nu = 1$, we obtain the classical relation

$$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d. \tag{3.3}$$

Since $\phi(n) \to \infty$ as $n \to \infty$ (see [8], p. 6, formula (16) for a more precise result), we see that the inverse transformation (3.2), if written in the form

$$\gamma_n = \sum_m b_{nm} s_m \qquad (n = 1, 2 \cdots), \tag{3.4}$$

has the properties $\sum_m b_{nm} = 1$, $b_{nm} \to 0$ as $n \to \infty$ for each fixed m. The question of the possible regularity of (3.4) will now be decided by the behavior of the sums

$$B_n = \sum_m |b_{nm}| = \frac{1}{\phi(n)} \sum_{d|n} \left|\mu\left(\frac{n}{d}\right)\right| d = \frac{1}{\phi(n)} \sum_{d|n} |\mu(d)| \frac{n}{d}.$$

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the canonical decomposition of n into primes we obtain

$$B_n = \frac{n}{\phi(n)} \sum_{d|n} |\mu(d)|/d = \frac{n}{\phi(n)} \left\{ 1 + \sum \frac{1}{p_1} + \sum \frac{1}{p_1 p_2} + \cdots + \frac{1}{p_1 p_2 \cdots p_k} \right\}$$

$$= \frac{n}{\phi(n)} \prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right) = \frac{n}{\phi(n)} \prod \left(1 - \frac{1}{p_i^2}\right) \Big/ \prod \left(1 - \frac{1}{p_i}\right)$$

and finally

$$B_n = \left(\frac{n}{\phi(n)}\right)^2 \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^2}\right).$$

Writing

$$\theta = \prod_{p} \left(1 - \frac{1}{p^2}\right), \qquad 0 < \theta < 1,$$

where p runs over all primes, we obtain

$$\theta \left(\frac{n}{\phi(n)}\right)^2 < B_n \leqq \left(\frac{n}{\phi(n)}\right)^2. \tag{3.5}$$

Observing that the sequence

$$n/\phi(n) = 1 \Big/ \prod_{i=1}^{k} (1 - p_i^{-1}) \qquad (n = 2, 3, \cdots)$$

is *not* bounded, we conclude from (3.5) that the sequence $\{B_n\}$ is likewise unbounded. By Toeplitz's theorem we conclude that *the transformation* (3.2) *is not regular.*

However, the inequalities (3.5) allow us to settle completely the question concerning the convergent *subsequences* of a ϕ-convergent sequence $\{\gamma_n\}$. We state the result as

THEOREM 2. *Let* $\{n_\nu\}$ *be a given sequence of increasing natural numbers. Then the assumption*

$$\phi\text{-}\lim \gamma_n = \lambda \tag{3.6}$$

implies that

$$\lim_{\nu \to \infty} \gamma_{n_\nu} = \lambda \tag{3.7}$$

if and only if the sequence $\{n_\nu\}$ *is such that*

$$\liminf_{\nu \to \infty} \frac{\phi(n_\nu)}{n_\nu} > 0. \tag{3.8}$$

Remarks. 1. The condition (3.8) is for instance satisfied if $\{n_\nu\}$ is the sequence of primes $\{p\}$; likewise if $\{n_\nu\} = \{2p\}$. For indeed, observe that

$$\frac{\phi(p)}{p} = 1 - \frac{1}{p} \to 1 > 0, \qquad \frac{\phi(2p)}{2p} = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{p}\right) \to \frac{1}{2} > 0$$

as $p \to \infty$. Thus (3.6) implies the relations

$$\gamma_p \to \lambda, \qquad \gamma_{2p} \to \lambda \text{ as the prime } p \to \infty. \tag{3.9}$$

2. Theorem 2 is of the nature of a "limitation theorem" (see [5], p. 95). Thus the sequence 1, 0, 1, 0, 1, 0, $\cdots$, which is $(C, 1)$-convergent to the limit $1/2$ is certainly *not* ϕ-convergent because for the present sequence

$$\gamma_p \to 1, \qquad \gamma_{2p} \to 0,$$

in contradiction to the necessary relations (3.9).

A proof of Theorem 2. Let us take from the system (3.2) or (3.4) only the relations corresponding to $n = n_\nu$:

$$\gamma_{n_\nu} = \sum_m b_{n_\nu, m} s_m, \qquad (\nu = 1, 2, \cdots). \tag{3.10}$$

The inequalities (3.5) which we write as

$$\left(\frac{\phi(n)}{n}\right)^2 \leqq \frac{1}{B_n} < \frac{1}{\theta}\left(\frac{\phi(n)}{n}\right)^2$$

show that the sequence $\{B_{n_\nu}\}$ is *bounded above*, or equivalently the sequence $\{1/B_{n_\nu}\}$ is bounded away from zero, if and only if the sequence $\{\phi(n_\nu)/n_\nu\}$ is *bounded away from zero.* Thus the transformation (3.10) is *regular* if and only if the condition (3.8) is verified. Therefore (3.8) is the necessary and sufficient condition for $s_n \to \lambda$ to imply $\gamma_{n_\nu} \to \lambda$ and our theorem is established.

4. A related notion of D-convergence of sequences. Let us again assume that $s_n \to \lambda$ which we express as above by writing

$$\phi\text{-lim } \gamma_n = \lambda. \tag{4.1}$$

Observe now that

$$\frac{\phi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right), \qquad (n = 1, 2, \cdots) \tag{4.2}$$

is a fraction which seems to roam freely in the range $[0, 1]$ as n ranges over the natural integers. By Theorem 2 the assumption (4.1) seems to imply that "essentially" $\gamma_n \to \lambda$, by which we mean that for any $\epsilon > 0$ the inequality $|\gamma_n - \lambda| < \epsilon$ holds for all n with the exception of some possibly infinite but "thin" sequence which depends on ϵ.

It is easy to make these vague ideas precise but requires some results con-

cerning the asymptotic behavior of the sequence (4.2), as $n\to\infty$, which are somewhat deeper than the tools which we have used so far. Let us start with the following definition (see *e.g.* [11], p. 316): If a sequence $S=\{n_\nu\}$ of increasing integers is given, we define

$$D^*(s) = \limsup_{n\to\infty} \frac{1}{n} \sum_{n_\nu \le n} 1, \qquad D_*(s) = \liminf_{n\to\infty} \frac{1}{n} \sum_{n_\nu \le n} 1,$$

and call them the *upper* and *lower* density of S, respectively. Clearly $0 \le D_*(S) \le D^*(S) \le 1$. If $D_*(S) = D^*(S)$ we say that the sequence S has a density whose value is

$$D(s) = \lim_{n\to\infty} \frac{1}{n} \sum_{n_\nu \le n} 1.$$

Evidently $D(S)=0$ if the sequence S is void or finite. For $n_\nu = 2\nu$ we find $D(S)=1/2$; for $n_\nu=\nu^2$, $D(S)=0$. More interesting examples for this concept are obtained as follows: Let

(4.3) $$x_1, x_2, x_3, \cdots$$

be a given sequence of reals and let us denote by $\{n; x_n \le t\}$ the (possibly void) sequence of integers n for which $x_n \le t$, where t is given real. If we now assume that the sequence (4.3) is asymptotically equidistributed in $[0, 1]$ (see [9], Part II, Ch. 4, or [7], Ch. 6) we find without difficulty that

(4.4) $$D\{n; x_n \le t\} = \begin{cases} 0 & \text{if} \quad t < 0, \\ t & \text{if} \quad 0 \le t \le 1, \\ 1 & \text{if} \quad t > 1. \end{cases}$$

Conversely, if (4.4) holds, we verify easily that the sequence (4.3) is equidistributed in $[0, 1]$. It is convenient to describe (4.4) by saying that the sequence (4.3) admits the *asymptotic distribution function*

(4.5) $$\omega_1(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ t & \text{if} \quad 0 \le t \le 1, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

The following generalization seems now natural. Let $\omega(t)$ be any given non-decreasing function defined for all real t and such that

$$\lim_{t\to-\infty} \omega(t) = 0, \qquad \lim_{t\to+\infty} \omega(t) = 1.$$

Functions $\omega(t)$ with these properties are called *distribution functions*. A sequence of reals (4.3) is said to admit the asymptotic distribution function $\omega(t)$ provided that the relation

$$D\{n; x_n \leqq t\} = \omega(t) \tag{4.6}$$

holds for every real t which is a point where $\omega(t)$ is continuous.

For example, the equidistribution in $[0, 1]$ of the sequence (4.3) is, by (4.4), a special case of the above definition for the special continuous distribution function (4.5). The particular sequence

$$x_n = \frac{\phi(n)}{n} \qquad (n = 1, 2, \cdots) \tag{4.7}$$

exhibits a somewhat different asymptotic behavior. We need the following

LEMMA 1. *The sequence* (4.7) *admits an asymptotic distribution function*

$$D\{n; \phi(n)/n \leqq t\} = \omega(t) \qquad (-\infty < t < \infty), \tag{4.8}$$

which is continuous for all real t.

A proof of this lemma can be found in the writer's dissertation [10], a simplified and more general version in [11]. Because $0 < x_n < 1$ for all n, it is clear that

$$\omega(t) = 0 \text{ if } t \leqq 0, \qquad \omega(t) = 1 \text{ if } t \geqq 1. \tag{4.9}$$

The nature of $\omega(t)$ in the range $[0, 1]$ does not concern us here; I may mention, however, that $\omega(t)$ increases strictly in the range $[0, 1]$ from $\omega(0) = 0$ to $\omega(1) = 1$ and that $\omega'(t) = 0$ for all values of t with the exception of a set of measure zero, as was shown by P. Erdös [3].

An almost immediate consequence of Lemma 1 is the following

LEMMA 2. *If the sequence* $S = \{n_\nu\}$ *has the property*

$$\lim_{\nu \to \infty} \frac{\phi(n_\nu)}{n_\nu} = 0 \tag{4.10}$$

then

$$D(S) = 0. \tag{4.11}$$

Indeed, let $S_t = \{n; \phi(n)/n \leqq t\}$ be the sequence appearing in (4.8). Now (4.10) clearly implies that the elements of the sequence $S = \{n_\nu\}$ are contained in the sequence S_t provided that $t > 0$ and that $\nu > \nu_0(t)$. But then evidently by (4.8),

$$D^*(S) \leqq D(S_t) = \omega(t).$$

Letting $t \to 0$ we have $\omega(t) \to \omega(0) = 0$ which shows that $D^*(S) = 0$ and hence (4.11) is established.

LEMMA 3. *If*

$$\phi\text{-}\lim \gamma_n = \lambda \tag{4.12}$$

and $\epsilon>0$ *then*

$$D\{n;\ |\gamma_n-\lambda|\ \geqq \epsilon\} = 0. \tag{4.13}$$

There is nothing to prove if the sequence defined by $|\gamma_n-\lambda|\geqq\epsilon$ is void or finite. Assuming this sequence to be infinite let us write

$$S(\epsilon) = \{n;\ |\gamma_n-\lambda|\ \geqq \epsilon\} = \{n_\nu\}_{\nu=1,2,\cdots}.$$

From the very definition of this sequence it is clear that $\gamma_{n_\nu}\not\to\lambda$. By Theorem 2 we conclude that

$$\liminf_{\nu\to\infty} \phi(n_\nu)/n_\nu = 0.$$

We claim that actually *the relation* (4.10) *holds*. For if not, then

$$\limsup_{\nu\to\infty} \phi(n_\nu)/n_\nu > 0$$

and we could therefore pick a subsequence $\{n_{\nu'}\}$, of $\{n_\nu\}$, for which

$$\liminf \phi(n_{\nu'})/n_{\nu'} > 0.$$

But now Theorem 2 would imply that $\gamma_{n_{\nu'}}\to\lambda$, which is absurd since

$$|\gamma_{n_{\nu'}}-\lambda|\ \geqq \epsilon \qquad \text{for } \textit{all } \nu'.$$

Thus (4.10) indeed holds and Lemma 2 implies the conclusion (4.13).

Lemma 3 suggests the following generalization of the concept of a convergent sequence:

DEFINITION OF D-CONVERGENCE. *We say that a sequence* $\{\gamma_n\}$ *is* D*-convergent to the* D*-limit* λ *provided that for every* $\epsilon>0$

$$D\{n;\ |\gamma_n-\lambda|\ \geqq \epsilon\} = 0, \tag{4.14}$$

in which case we write

$$D\text{-}\lim \gamma_n = \lambda. \tag{4.15}$$

Clearly $\lim \gamma_n=\lambda$ implies (4.15) since void or finite sequences have vanishing densities. In terms of this definition we may restate Lemma 3 as

THEOREM 3. *If* $\phi\text{-}\lim \gamma_n=\lambda$, *then also* $D\text{-}\lim \gamma_n=\lambda$.

The converse is evidently false. Thus if

$$\gamma_n = \begin{cases} 1 & \text{if } n \text{ is a prime,} \\ 0 & \text{otherwise,} \end{cases}$$

then $D\text{-}\lim \gamma_n=0$ because the density of the sequence of primes is zero. On the other hand the relations

$$\gamma_p \to 1, \qquad \gamma_{2p} \to 0$$

hold and rule out the ϕ-convergence of $\{\gamma_n\}$ in view of the necessary conditions (3.9).

We conclude this section with a few obvious properties of D-convergence whose simple proofs may be omitted:

1. D-lim $x_n=\xi$, D-lim $x_n=\eta$ imply $\xi=\eta$.
2. D-lim $x_n=\xi$ implies D-lim $(cx_n)=c\xi$.
3. D-lim $x_n=\xi$, D-lim $y_n=\eta$ imply D-lim $(x_n+y_n)=\xi+\eta$.

5. A criterion for D-convergence. The thoughtful reader has surely noticed the close connection of D-convergence with the notion of an asymptotic distribution function $\omega(t)$ as described by the relation (4.6). Indeed, let us consider the special distribution function

$$\omega_0(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geqq 1. \end{cases} \tag{5.1}$$

A moment's reflection will show that

$$D\text{-lim } x_n = \xi$$

if and only if the sequence $\{x_n\}$ *admits the asymptotic distribution function* $\omega_0(t-\xi)$.

Thus D-convergence, like equidistribution (see (4.4), (4.5)), is equivalent to the sequence admitting a certain special remarkable distribution function, in this case the function (5.1) or its translates $\omega_0(t-\xi)$.

The kinship of D-convergence to equidistribution raises the question whether there is also for D-convergence a criterion analogous to Weyl's criterion for equidistribution (see [7], p. 76 or [9], Problem 165, p. 70). The answer is affirmative and may be stated as

THEOREM 4. *For*

$$D\text{-lim } x_n = \xi \tag{5.2}$$

it is necessary and sufficient that

$$\lim_{n\to\infty} \frac{1}{n} (e^{itx_1} + e^{itx_2} + \cdots + e^{itx_n}) = e^{it\xi} \text{ for every real } t. \tag{5.3}$$

Proof of necessity. This will readily follow from a few simple observations which we state as lemmas.

LEMMA 4. *If* (5.2) *holds and*

$$|x_n| < K \quad \text{for all } n, \tag{5.4}$$

then

$$(C, 1)\text{-}\lim x_n = \xi. \tag{5.5}$$

Indeed, without loss of generality we may assume that $\xi=0$. This means that if $\epsilon>0$ and if we denote by N_n the number of ν among $\nu=1, \cdots, n$ for which $|x_\nu| \geqq \epsilon$, then

$$\lim_{n\to\infty} \frac{N_n}{n} = 0. \tag{5.6}$$

Now

$$\left|\frac{x_1 + \cdots + x_n}{n}\right| \leqq \frac{N_n K + (n - N_n)\epsilon}{n} = \epsilon + (K - \epsilon)\frac{N_n}{n}$$

where, by (5.6), the right side is $<2\epsilon$ provided that n is sufficiently large. Thus (5.5) is established.

LEMMA 5. *If* (5.2) *holds and* $g(x)$, *defined for all real* x *is continuous at* $x=\xi$, *then*

$$D\text{-}\lim g(x_n) = g(\xi). \tag{5.7}$$

Indeed, to an $\epsilon>0$ corresponds a $\delta>0$ such that

$$|x - \xi| < \delta \text{ implies } |g(x) - g(\xi)| < \epsilon.$$

But then $|g(x)-g(\xi)| \geqq \epsilon$ implies $|x-\xi| \geqq \delta$ and in particular $|g(x_n)-g(\xi)| \geqq \epsilon$ implies $|x_n-\xi| \geqq \delta$. Thus $\{n; |g(x_n)-g(\xi)| \geqq \epsilon\} \subset \{n; |x_n-\xi| \geqq \delta\}$ and therefore

$$D^*\{n; |g(x_n) - g(\xi)| \geqq \epsilon\} \leqq D\{n; |x_n - \xi| \geqq \delta\} = 0$$

because of (5.2). This proves (5.7).

The *necessity* of (5.3) is now clear. Indeed, for a *fixed* value of t, e^{itx} is a *continuous* function of x. By Lemma 5 (5.2) implies

$$D\text{-}\lim_{n\to\infty} e^{itx_n} = e^{it\xi}. \tag{5.8}$$

Since $\{e^{itx_n}\}$ is a *bounded* sequence, (5.8) implies, by Lemma 4, that (5.3) holds.

Proof of sufficiency. This proof requires some integral calculus. Let us define a continuous function $M(x)$ by

$$M(x) = \begin{cases} 0 & \text{if } x < -1, \\ 1 + x & \text{if } -1 \leqq x < 0, \\ 1 - x & \text{if } 0 \leqq x < 1, \\ 0 & \text{if } x \geqq 1. \end{cases} \tag{5.9}$$

We take for granted the fact that $M(x)$ allows the integral representation

$$M(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t/2}{t/2}\right)^2 e^{itx}dt \qquad (-\infty < x < \infty), \tag{5.10}$$

as can be proved, for instance, by means of Cauchy's calculus of residues. We observe first that it suffices to show that (5.3) implies (5.2) for the case when $\xi=0$. For if this is already known, as (5.3) implies

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{it(x_1-\xi)}+\cdots+e^{it(x_n-\xi)}\right)=1,$$

we may conclude that $D\text{-lim}\,(x_n-\xi)=0$. But $D\text{-lim}\,\xi=\xi$. Adding the two results we obtain the desired conclusion (5.2).

Let us now assume

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{itx_1}+\cdots+e^{itx_n}\right)=1 \qquad \textit{for every real } t, \tag{5.11}$$

and let us prove that

$$D\{n;|\,x_n|\geqq\epsilon\}=0 \qquad \textit{if } \epsilon>0. \tag{5.12}$$

By an appropriate change of variables (5.10) furnishes

$$M(x/\epsilon)=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon t/2}{\epsilon t/2}\right)^2 e^{itx}dt \tag{5.13}$$

whence

$$\frac{1}{n}\sum_{\nu=1}^{n}M(x_\nu/\epsilon)=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon t/2}{\epsilon t/2}\right)^2\left\{\frac{1}{n}\sum_{\nu=1}^{n}e^{ix_\nu t}\right\}dt.$$

Observe now that (5.13) is an *absolutely* convergent integral, *i.e.*, the relation is valid with the integral taken in the sense of Lebesgue. Moreover (5.11) holds while

$$\left|\frac{1}{n}\sum_{\nu=1}^{n}e^{ix_\nu t}\right|\leqq 1 \qquad \text{for all real } t, \text{ all } n.$$

By the bounded convergence theorem we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}M(x_\nu/\epsilon)=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon t/2}{\epsilon t/2}\right)^2 dt=M(0)=1,$$

or

$$\lim\frac{1}{n}\sum_{\nu=1}^{n}M(x_\nu/\epsilon)=1. \tag{5.14}$$

It is clear that a somewhat longer argument by elementary method may replace

our reference to Lebesgue integration theory. If we let N_n be the number of solutions of $|x_\nu| \geqq \epsilon$ among $\nu = 1, \cdots, n$, then clearly, by (5.9), we have

$$\sum_{\nu=1}^{n} M(x_\nu/\epsilon) \leqq n - N_n$$

or

$$\frac{N_n}{n} \leqq 1 - \frac{1}{n} \sum_{\nu=1}^{n} M(x_\nu/\epsilon).$$

But the right-hand side $\to 0$ as $n \to \infty$, in view of (5.14). Thus $N_n/n \to 0$ and (5.12) is established. This completes a proof of our theorem.

6. A remark by Hans Rademacher. I owe to Hans Rademacher the following interesting remark which is published here with his kind permission. Years ago he already considered the function $f(x)$ defined by our relation (1.2) and found the following

THEOREM 5 (Rademacher). *If the sequence* (1.1) *satisfies the condition*

$$\lim_{n \to \infty} n^k \gamma_n = 0 \qquad (k \text{ an integer} \geqq 2) \tag{6.1}$$

then the function $f(x)$ *defined by* (1.2) *is differentiable at* $x = \xi$, *where* ξ *is an algebraic irrationality of degree not exceeding* k.

We may indeed, assume that ξ is an algebraic number of degree k. A famous theorem of J. Liouville (see [7], pp. 90–91) insures then the existence of a constant $c = c(\xi) > 0$ such that

$$\left| \frac{p}{q} - \xi \right| > \frac{c}{q^k}$$

for any rational fraction p/q, $(q > 0)$. By (1.2) we now have

$$\left| \frac{f(p/q) - f(\xi)}{p/q - \xi} \right| \leqq \frac{1}{c} q^k \gamma_q.$$

Thus, if $p/q \to \xi$, hence $q \to \infty$, our assumption (6.1) implies that $f'(\xi) = 0$.

In particular, if for instance $\gamma_n = 2^{-n}$, then (6.1) holds for all k and $f(x)$ is differentiable at all algebraic irrationalities.

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HEURISTIC REASONING IN THE THEORY OF NUMBERS

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A deep but easily understandable problem about prime numbers is used in the following to illustrate the parallelism between the heuristic reasoning of the mathematician and the inductive reasoning of the physicist. The experts may judge whether the parallelism is more serious than the tone of presentation which is adapted to a wider audience.

1. "Till now the mathematicians tried in vain to discover some order in the sequence of the prime numbers and we have every reason to believe that there is some mystery which the human mind shall never penetrate. To convince oneself, one has only to glance at the tables of primes which some people took the trouble to compute beyond a hundred thousand, and one perceives that there is no order and no rule. This is so much more surprising as the arithmetic gives us definite rules with the help of which we can continue the sequence of the primes as far as we please, without noticing, however, the least trace of order."*

So wrote Euler about two centuries ago, yet the prime numbers may inspire the contemporary mathematician with the same feeling of mystery that Euler so vividly expressed. The primes remain puzzling in spite of many important discoveries made in the meantime. Let us look at some of these discoveries.

The intervals between successive primes are irregular, but these intervals seem to become larger "on the whole" (the primes seem to become scarcer) as we proceed in the sequence of numbers. Since Euler's time a definite law of this phenomenon was discovered (conjectured by Legendre and Gauss, investigated by Chebyshev and Riemann, finally proved by Hadamard and de la Vallée Poussin, proved recently in an essentially different "elementary" manner by Atle Selberg and Paul Erdös). We may formulate this law, the "prime number theorem," intuitively although not quite precisely, as follows: The probability

* See L. Euler, Opera Omnia, ser. 1, vol. 2, p. 241 or G. Pólya, Mathematics and Plausible Reasoning, Princeton, vol. 1, p. 91.

that a large integer x should be a prime, is $1/\log x$ (where $\log x$ is the natural logarithm of x).*

The following short table exhibits the first primes (with two exceptions) classified according to their last digit.

	11		31	41		61	71			101	
3	13	23		43	53		73	83		103	113
7	17		37	47		67			97	107	
	19	29			59		79	89		109	

If we set apart 2 and 5, the prime factors of 10, the last figure in the decimal symbol of a prime cannot be 0, 2, 4, 5, 6, or 8 (since neither 2 nor 5 should be a divisor) and must, therefore, be 1, 3, 7, or 9. Thus, with respect to ten (*modulo* 10) there are four kinds of primes which are listed in the four horizontal lines of the foregoing table, respectively. Since Euler's time, a general law has been discovered (most of the credit for its discovery is due to Dirichlet) which, applied to our particular case, asserts that there are infinitely many prime numbers of each kind and, what is more, that each kind is equally probable. Therefore, in an extensive table of prime numbers there must be roughly as many primes ending with 1 as primes ending with 3.

Euler mentions a table of primes that goes beyond 10^5. Since his time much more extensive tables have been computed, especially in the last decade with the help of machines. Data derived from these tables may suggest problems not yet considered by Euler.

2. The least possible distance between two consecutive primes is 2, if we set apart the unique case of the primes 2 and 3. Two primes having this minimum distance are called *twin primes*. Here is a list of the twin primes under 100:

3, 5 5, 7 11, 13 17, 19 29, 31 41, 43 59, 61 71, 73

We can generalize this situation and consider a prime p that is escorted at a given distance d by another prime $p'=p+d$. (This situation is uninteresting unless d is even; we do not care whether there are or are not other primes between p and p'.) Here is a list of all such pairs at the distance 6, in which the

* The irregular distribution of primes ("there is no order and no rule") strongly suggests the idea of probability and chance. Yet this is paradoxical: Whether any given integer is a prime or not, can be decided by the "definite rules" of arithmetic—where and how could chance enter the picture? The paradox can be somewhat explained (or deepened) by a physical analogy. The kinetic theory of matter considers the probability distribution of the velocities of the molecules in a gas. Yet this is paradoxical: The velocities resulting from the collision of two molecules can be exactly predicted from the data of the collision by the "definite rules" of classical deterministic mechanics —where and how could chance enter the picture? The determinateness of the simple single event and the probabilistic theory of the highly composite whole may seem to be equally compatible (or incompatible) in both cases.

first prime does not (but its escort may) exceed 100:

5, 11	7, 13	11, 17	13, 19	17, 23	23, 29	31, 37	37, 43
41, 47	47, 53	53, 59	61, 67	67, 73	73, 79	83, 89	97, 103

It is curious that the second kind of pairs is more numerous. We count 8 pairs of twin primes and exactly twice as many pairs of primes at the distance 6. Let us take now instead of 10^2 the considerably higher bound $3\cdot10^7$. Under thirty million there are 152892 primes followed by another prime at the distance 2, but nearly twice as many, namely 304867 primes followed by another at the distance 6.

The numbers of these prime pairs have been obtained by Professor and Mrs. D. H. Lehmer with the use of appropriate computing apparatus; they computed, up to the same limit $3\cdot10^7$, the number of primes escorted by another prime at the distance d for $d=2, 4, 6, 8, \cdots, 70$. I wish to thank them here for their kind permission to use their interesting material. I wish to use some of their results to offer the unprejudiced reader a particularly suitable opportunity for an inductive investigation in pure mathematics.

It will be convenient to introduce here some notation. Let $\pi_d(x)$ stand for the number of those prime numbers p that satisfy two conditions:

$$p \leqq x, \qquad p + d \text{ is a prime number.}$$

For instance,

$$\pi_2(100) = 8\ , \qquad \pi_2(30\ 000\ 000) = 152892,$$
$$\pi_6(100) = 16, \qquad \pi_6(30\ 000\ 000) = 304867.$$

I set

$$\pi_d(3\cdot10^7)/\pi_2(3\cdot10^7) = R_d.$$

For instance, $R_6 = 304867/152892 = 1.9940$, approximately. A small part of the material computed by Professor and Mrs. Lehmer is collected in Table I.

d	R_d	12	1.9985	24	1.9976	36	1.9997	48	1.9965	60	2.6632
2	1.0000	14	1.1985	26	1.0910	38	1.0566	50	1.3308	62	1.0341
4	0.9979	16	1.0001	28	1.1974	40	1.3330	52	1.0892	64	0.9999
6	1.9940	18	1.9982	30	2.6632	42	2.3987	54	1.9981	66	2.2186
8	0.9996	20	1.3311	32	0.9970	44	1.1097	56	1.1957	68	1.0663
10	1.3317	22	1.1088	34	1.0645	46	1.0467	58	1.0349	70	1.5977

TABLE I. VALUES OF R_d

3. Now, let us start our inductive research. At any moment at which the reader feels inspired, he should interrupt the reading and try to guess the result by himself.

The four kinds of prime numbers that we have considered in Section 1 (ending with 1, 3, 7 or 9 in the decimal notation, respectively) are known to be equally frequent. Are the 35 kinds of prime numbers with which Table I is concerned also equally frequent? If it were so, all the ratios R_d contained in Table I should be approximately equal to one. In fact, remarkably enough, a few entries in Table I are pretty close to the value 1, but the majority seem to deviate significantly from 1. The analogy with the previous case does not seem to go far. Yet, perhaps, the analogy holds at least in one respect: the ratio $\pi_d(x)/\pi_2(x)$ may converge towards some limit (not necessarily 1) when x tends to infinity, and the ratio $R_d=\pi_d(3\cdot 10^7)/\pi_2(3\cdot 10^7)$ entered into Table I may be an approximation to that limit.

We face here a situation somewhat analogous to the situation that the chemists faced around 1800 when they were about to discover the Law of Multiple Proportions. They had to perceive behind their experimental data distorted by unavoidable errors of observation the ratios of simple multiples of the atomic weights, and we have to perceive behind the approximate ratios R_d collected in Table I the true limiting ratios. To guess these limiting ratios is a challenging task.

We have already observed that some values of R_d are very close to 1; they correspond to $d=2$, 4, 8, 16, 64. (For $d=2$ the value is exactly 1, but this is trivial.) We can scarcely fail to notice here the powers of 2. By the way, these values of R_d so close to 1 are also the smallest values in the table. Are there other entries in the table so nearly equal to each other?

In trying to answer this question we may notice that the entries corresponding to

$$d = 6,\ 12,\ 24,\ 48$$

are approximately equal to each other, and so are those corresponding to

$$d = 10,\ 20,\ 40$$

or those corresponding to

$$d = 14,\ 28,\ 56.$$

In general, multiplication of d by 2 seems to leave the value of R_d almost unchanged.

What about multiplication by 3? It approximately doubles the value of R_d in certain transitions, as from

2 to 6,	4 to 12,	8 to 24,	16 to 48,
10 to 30,	20 to 60,	14 to 42,	22 to 66.

Yet it is not so in other cases, as

$$6 \text{ to } 18, \quad 12 \text{ to } 36, \quad 18 \text{ to } 54;$$

in these latter cases the multiplication of d by 3 leaves the value of R_d almost unchanged. How can you account for this different behavior?

And so on, from question to question, by observation and tentative generalization, carefully checking each guess, the reader may discover that many of the values R_d contained in Table I come very close to simple fractions; see Table II.

d	2 4 8	16 32 64	6 12 18 24	36 48 54	10 20 40 50	14 28 56	22 44	30 60	42	66	70
R_d (approx.)	1		$\frac{2}{1}$		$\frac{4}{3}$	$\frac{6}{5}$	$\frac{10}{9}$	$\frac{8}{3}$	$\frac{12}{5}$	$\frac{20}{9}$	$\frac{8}{5}$

TABLE II. SIMPLE APPROXIMATIONS TO SOME R_d

Table II strongly suggests that R_d *depends only on the decomposition of* d *into prime factors.* More precisely, just the presence of a prime factor in, or its absence from, the decomposition seems to be relevant; for instance, to all values of d of the form $2^\alpha 3^\beta$ with $\alpha, \beta = 1, 2, 3, \cdots$ there corresponds the same value of R_d (approximately).

Moreover, to each prime factor of d there seems to correspond a factor of R_d; to the (unavoidable) factor 2 of d, the (trivial) factor 1 of R_d; to the prime factors

$$3, \quad 5, \quad 7, \quad 11$$

of d, the following factors of R_d:

$$\frac{2}{1}, \quad \frac{4}{3}, \quad \frac{6}{5}, \quad \frac{10}{9},$$

respectively. Then, when d is a product of different primes (or powers of different primes) R_d seems to be the product of the corresponding factors.

4. All such observations point to the (conjectural) formula

$$(1) \qquad \pi_d(x) \sim \pi_2(x) \prod_{p|d} \frac{p-1}{p-2},$$

where the product $\prod_{p|d}$ is extended over all different *odd* prime factors p of the even number d.* The sign $\sim$ can be interpreted either vaguely or strictly. In a

* The usual abbreviation $a|b$ means "a divides b" or "a is a divisor of b." We shall need later also the abbreviation $a \nmid b$ which means "a is not a divisor of b."

vague interpretation $\sim$ means "approximately equal;" in the strict sense it means "the ratio of the two sides tends to 1 when x tends to ∞." The formula is merely a conjecture which we can conceive quite naively by examining Table I. In Table III, the observed values of R_d, taken from Table I and styled now R_d (obs.), are compared with the corresponding conjectural limiting values, styled R_d (theor.). This comparison yields strong inductive evidence for the conjecture which could be further strengthened by use of other data computed by Professor and Mrs. Lehmer.

d	R_d (obs.)	R_d (theor.)	24	1.9976	2.0000	48	1.9965	2.0000
2	1.0000	1.0000	26	1.0910	1.0909	50	1.3308	1.3333
4	0.9979	1.0000	28	1.1974	1.2000	52	1.0892	1.0909
6	1.9940	2.0000	30	2.6632	2.6667	54	1.9981	2.0000
8	0.9996	1.0000	32	0.9970	1.0000	56	1.1957	1.2000
10	1.3317	1.3333	34	1.0645	1.0667	58	1.0349	1.0370
12	1.9985	2.0000	36	1.9997	2.0000	60	2.6632	2.6667
14	1.1985	1.2000	38	1.0566	1.0588	62	1.0341	1.0345
16	1.0001	1.0000	40	1.3330	1.3333	64	0.9999	1.0000
18	1.9982	2.0000	42	2.3987	2.4000	66	2.2186	2.2222
20	1.3311	1.3333	44	1.1097	1.1111	68	1.0663	1.0667
22	1.1088	1.1111	46	1.0467	1.0476	70	1.5977	1.6000

TABLE III. VALUES OF R_d, OBSERVED AND "THEORETICAL"

5. We have before us a precise, general, but enigmatic formula derived from, and quite well verified by, observations. Of course, we wish to understand it, we wish to explain it. When we are looking at it, our situation is similar to that of Newton looking at the laws of Kepler or to that of Niels Bohr looking at Balmer's formula. The word "similar" must be correctly understood. Similar figures may be very different in magnitude, but they show the same proportions, and so do in a sense the three situations we have just compared.

We wish to explain that conjectural formula about prime numbers. Both the irregular distribution of the primes and the structure of the conjectural formula strongly suggest an explanation by probability. I wish to present such an explanation. We shall arrive at it in two steps (of which the second is much more dangerous).

PROBLEM I. *Let p denote a given prime number, d a given integer, and x a large integer chosen at random. Find the probability that neither x nor $x+d$ is divisible by p.*

The reader may visualize the integers as successive intervals of equal length along an infinite straight line, some sort of super-roulette. The interval is red or green, according as the integer is, or is not, divisible by p; among any p consecutive intervals there is always just one that is red. A ball is rolled along the line and steps in the interval x.

We have to distinguish two cases.*

First case: $p \mid d$. In this case $x+d$ falls on a multiple of p (a red space) if, and only if, x itself falls on such a multiple. Therefore, out of any p consecutive numbers (spaces), $p-1$ are favorable (green) and so the required probability is $(p-1)/p$.

Second case: $p \nmid d$. Even if x does not fall on a multiple of p, $x+d$ may. Therefore, out of any p consecutive numbers just $p-2$ are favorable. The required probability is $(p-2)/p$.

PROBLEM II. *Let d denote a given even integer, and x a large integer chosen at random. Find the probability P_d that both x and $x+d$ are prime numbers.*

In order that both x and $x+d$ should be prime numbers, a sequence of conditions must be satisfied:

First, neither x nor $x+d$ is divisible by 2;

then, neither x or $x+d$ is divisible by 3;

then, neither x nor $x+d$ is divisible by 5;

and so on. The general form of this condition is: neither x nor $x+d$ is divisible by p where p is a prime number.

We have computed above the probability for the fulfillment of any single one of these conditions. Now we have to compute the probability that all these conditions are fulfilled at the same time, all these events are realized simultaneously.

Two difficulties arise here: Are these events independent? How far should we go with p? In fact, these two difficulties may be connected, but at this stage of the game it will be better not to examine them too thoroughly; let us now proceed quickly and see whether anything worthwhile turns up.

Are the events independent? We do not know, but let us assume it. Also the physicist is inclined to assume the independence of the probabilities he deals with—not because he knows that they are independent, but interdependent probabilities are so much more difficult to handle—and so let us assume independence in our case too, although we have no better reasons than the physicist.

Having made this assumption all we have to do is to multiply probabilities computed above. We distinguish three cases:

* For the symbols $\mid$ and $\nmid$, see footnote p. 379.

$p=2$ (which is a divisor of the even number d);
p is odd and is a divisor of d;
p is odd and is not a divisor of d.

Accordingly, the required probability P_d is a product of three kinds of factors:

$$P_d = \frac{1}{2} \prod_{p|d} \frac{p-1}{p} \prod_{p \nmid d} \frac{p-2}{p}. \tag{2}$$

In this formula (2) (and in the following formulas (3), (4)) the letter p stands for an *odd* prime number.

How far should we go with p? Of course, on the right hand side of formula (2) we extend the first product over all odd prime factors of the given number d. In the second product, we take all the odd primes not dividing d up to a certain large upper bound, depending on the considered large number x—but let us *postpone* the decision, how far to go, how large that upper bound should precisely be.

We can transform formula (2) as follows:

$$P_d = \prod_{p|d} \frac{p-1}{p-2} \cdot \frac{1}{2} \cdot \prod_{p} \frac{p-2}{p}; \tag{3}$$

the second product on the right hand side of (3) is extended over *all* odd primes p under a certain (large, but not yet definitely characterized) upper bound. The first product is extended over the odd prime divisors of d; if d happens to be 2 (or a power of 2) there are no odd prime divisors, that first product is empty, and has to be replaced by 1. Therefore

$$P_d = \prod_{p|d} \frac{p-1}{p-2} \cdot P_2. \tag{4}$$

Yet the ratio of the probabilities P_d/P_2 should be approximately the same as the ratio of the observed numbers $\pi_d(x)/\pi_2(x)$—and so the formula (4) just derived justifies the conjectural formula (1)—complete success!

6. Unfortunately, our reasoning is vulnerable and the success is illusory. We left a gap in our derivation (we did not decide how far to go with p) and if we try to fill this gap, we run into trouble. The trouble becomes manifest if we try to apply our reasoning to the simplest analogous problem, the result of which is well known.

PROBLEM III. *Find the probability that x, a large integer chosen at random, is a prime number.*

By reasoning as we did in solving Problem I and assuming the independence of the probabilities involved as we did in solving Problem II we obtain the answer $\prod(p-1)/p$; the product is extended to all primes p not surpassing a cer-

tain bound—but what should be the bound? The number x is certainly prime if it is not divisible by any prime $p<x$. This leads to the evaluation of the desired probability

$$\prod_{p<x} \frac{p-1}{p} \sim \frac{\mu}{\log x} \tag{5}$$

where $\mu=0.561459\cdots=e^{-c}$ and $c=0.577215\cdots$ is the familiar constant of Mascheroni and Euler; the asymptotic evaluation in (5) (on the right hand side of the sign $\sim$) which is valid for $x\to\infty$, is due to Mertens.*

Now, the value (5) is too small. The probability in question is known to be $1/\log x$; this is just the prime number theorem. And we can "explain" somehow why the result is wrong: If the integer x is not divisible by any prime p which does not exceed $x^{1/2}$, x itself must be a prime—and so divisibility by primes exceeding $x^{1/2}$ is, in fact, *not* independent of the smaller primes.

Let us try to modify (5) by considering only primes p not exceeding $x^{1/2}$. This leads us to

$$\prod_{p\le x^{1/2}} \frac{p-1}{p} \sim \frac{\mu}{\log (x^{1/2})} = \frac{1.122\cdots}{\log x} \tag{6}$$

(we used Mertens' result (5)) and this value is too large.

Let us, however, imitate the physicists who, without hesitation, modify their theories to fit the observed facts. And so let us do a thing between (5) and (6) and extend the product to all *primes not exceeding* x^{μ}. We obtain so

$$\prod_{p<x^{\mu}} \frac{p-1}{p} \sim \frac{1}{\log x}, \tag{7}$$

the right result.

I do not pretend to understand why the introduction of the upper bound x^{μ} *should* yield the right result. For that matter, when the quanta were introduced, no physicist pretended to understand why energy should be obtainable (as salt or sugar is in the self-service store) only in uniform little packages, in multipla of a certain unit. Yet the criterion of a physical theory is its applicability. Let us apply the (unintelligible) trick that gave us the right expression for the prime number theorem to our formula (3). Extending the second product to *odd* primes p inferior to x^{μ}, we are led to

$$\begin{aligned} P_d &= \prod_{p\mid d} \frac{p-1}{p-2}\cdot\frac{1}{2}\prod_{p<x^{\mu}} \frac{p-2}{p} \\ &\sim \prod_{p\mid d} \frac{p-1}{p-2}\cdot 2\prod_{p<x^{\mu}} \frac{(p-2)p}{(p-1)^2}\,\frac{1}{(\log x)^2}; \end{aligned} \tag{8}$$

* *Cf.* G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1938, p. 349, Th. 430.

we have used Mertens' result (5). It is easily seen that (8) is equivalent to

$$P_d \sim 2C_2 \prod_{p|d} \frac{p-1}{p-2} \frac{1}{(\log x)^2}, \tag{9}$$

where C_2 stands for the convergent infinite product

$$\Pi\left(1 - \frac{1}{(p-1)^2}\right)$$

extended to all odd primes $p=3, 5, 7, 11, \cdots$. The asymptotic formula (9) is due to Hardy and Littlewood, yet even their argument, which is incomparably deeper and more difficult than the one presented here, does not prove (9); it just confers on (9) another kind of plausible evidence. Yet all available numerical data also seem to support (9).

Let us recall that we have attained (9) by combining two analogies, one of which was extremely "natural" and the other (the "trick of the magic μ") extremely "artificial." And let us try to draw the moral: mathematicians and physicists think alike; they are led, and sometimes misled, by the same patterns of plausible reasoning.*

A CHAIN OF CYCLIC GROUPS

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1. Introduction. Consider the chain of groups $\mathfrak{G}_0, \mathfrak{G}_1, \cdots, \mathfrak{G}_i, \cdots$ where $\mathfrak{G}_0$ is a cyclic group of order m, and $\mathfrak{G}_i$ is the automorphism group of $\mathfrak{G}_{i-1}$, $i=1, 2, \cdots$. We ask when the chain consists entirely of cyclic groups. Obviously, when $m=1$ this will be so, and we suppose henceforth that $m>1$.

When $\mathfrak{G}_0$ is cyclic of order m with generator a it is well known that its automorphism group, $\mathfrak{G}_1$, is of order $t=\phi(m)$ and that $\mathfrak{G}_1$ is isomorphic to the multiplicative group modulo m of integers less than and relatively prime to m.

* See G. H. Hardy and J. E. Littlewood, Some problems of "Partitio numerorium": On the expression of a number as a sum of primes, Acta Math., vol. 44, 1922, pp. 1–70, especially Conjecture B on p. 42. The more general conjecture on p. 61 (Theorem X 1) is also obtainable by the foregoing reasoning. See also the literature quoted (and criticized) on pp. 32–34, especially the writings of Sylvester, concerning the use of probabilities in questions of similar nature. The crux of the matter may be so expressed: When we consider a fixed number of primes, the "probabilities" introduced can be regarded as "independent," but they cannot be so regarded when the number of primes considered increases in an arbitrary manner. (*Added in proof.* Profesor E. M. Wright drew my attention to a paper by the late Lord Cherwell in the Quart. J. Math., vol. 17, 1946, pp. 46–62, which has a certain contact with the present paper, and to a paper by Lord Cherwell and himself which is scheduled to appear in a coming volume of the Quarterly.)

2. $\mathfrak{G}_0$, $\mathfrak{G}_1$, $\mathfrak{G}_2$, and $\mathfrak{G}_3$. We write

$$\mathfrak{G}_1 = [1, a_2, \cdots, a_t],$$

where $1, a_1, a_2, \cdots, a_t$ are the $\phi(m)$ integers less than and relatively prime to m. Then, since $\mathfrak{G}_1$ must be cyclic, there exists an a_i such that $\phi(m)$ is the smallest exponent s with the property that $a_i^s \equiv 1 \pmod{m}$. That is, one of the a_i must be a primitive root of m. But this will occur* if and only if $m = p_0^{n_0}$, $m = 2p_0^{n_0}$, $m = 2$, or $m = 4$. ($p_0, p_1, p_2, \cdots$ will always denote an odd prime.) Under these circumstances, $\mathfrak{G}_2$ will be isomorphic to the multiplicative group mod $\phi(m)$ of order $\phi^2(m)$ of integers less than and relatively prime to $\phi(m)$.

If $m \neq 2$ or 4 and $\mathfrak{G}_2$ is also cyclic, we must have either

(1) $\phi(m) = 2$; (2) $\phi(m) = 4$; (3) $\phi(m) = p_1^{n_1}$; or (4) $\phi(m) = 2p_1^{n_1}$.

Case 1. $\phi(m) = 2$. Then $\phi(m) = p_0^{n_0-1}(p_0 - 1) = 2$, $p_0 = 3$, $n_0 = 1$, $m = 3$ or $2 \cdot 3 = 6$.

Case 2. $\phi(m) = 4$. Then $\phi(m) = p_0^{n_0-1}(p_0 - 1) = 4$ so that $p_0 = 5$, $n_0 = 1$, and $m = 5$ or $2 \cdot 5 = 10$.

Case 3. $\phi(m) = p_1^{n_1}$. Then $p_0^{n_0-1}(p_0 - 1) = p_1^{n_1}$, which is impossible.

Case 4. $\phi(m) = 2p_1^{n_1}$. Then $p_0^{n_0-1}(p_0 - 1) = 2p_1^{n_1}$.

(a) If $p_0 = p_1$, $p_0 = 3$, $n_1 = n_0 - 1$, $m = 3^{n_0}$ or $2 \cdot 3^{n_0}$.

(b) If $p_0 \neq p_1$, $n_0 = 1$, $m = p_0 = 2p_1^{n_1} + 1$ or $m = 2p_0$.

We continue, then, with Case 4b, where

$$m = p_0 = 2p_1^{n_1} + 1 \text{ or } m = 2p_0. \quad \text{Then} \quad \phi(m) = p_0 - 1 = 2p_1^{n_1}$$

and demand further that

$$\phi^2(m) = \phi(2p_1^{n_1}) = p_1^{n_1-1}(p_1 - 1)$$

be equal to either

$$(1)\ 2; \quad (2)\ 4; \quad (3)\ p_2^{n_2}; \quad \text{or} \quad (4)\ 2p_2^{n_2}.$$

Case 1. $p_1 = 3$, $n_1 = 1$, $m = 2 \cdot 3 + 1 = 7$ or $2 \cdot 7 = 14$.

Case 2. $p_1 = 5$, $n_1 = 1$, $m = 2 \cdot 5 + 1 = 11$ or $2 \cdot 11 = 22$.

Case 3. Impossible.

Case 4. (a) $p_1 = p_2$, $p_1 = 3$, $n_2 = n_1 - 1$, $m = p_0 = 2 \cdot 3^{n_1} + 1$ or $m = 2p_0$ where, then, $2 \cdot 3^{n_1} + 1$ must be a prime.

(b) $p_1 \neq p_2$, $n_1 = 1$, $p_1 = 2p_2^{n_2} + 1$, $m = p_0 = 2p_1^{n_1} + 1 = 2(2p_2^{n_2} + 1) + 1 = 4p_2^{n_2} + 3$ or $m = 2p_0$.

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* See, for example, L. E. Dickson, Introduction to the Theory of Numbers, Chicago, 1929, p. 20.

3. The induction. We now perform an induction and investigate the passage from $\phi^{k-1}(m)$ to $\phi^k(m)$ where we suppose that $p_{k-1} \neq 3$ or 5 so that the process has not terminated. Thus, we assume that $p_{k-2} = 2p_{k-1}^{n_{k-1}} + 1$,

$$\phi^k(m) = p_{k-1}^{n_{k-1}}(p_{k-1} - 1) \quad \text{and} \quad n_{k-2} = 1$$

and demand that this be equal to

$$(1)\ 2; \qquad (2)\ 4; \qquad (3)\ p_k^{n_k}; \quad \text{or} \quad (4)\ 2p_k^{n_k}.$$

Case 1. $p_{k-1} = 3$, $n_{k-1} = 1$.

Case 2. $p_k - 1 = 5$, $n_{k-1} = 1$.

Case 3. Impossible.

Case 4. (a) $p_{k-1} = 3$.
(b) $n_{k-1} = 1$, $p_{k-1} = 2p_k^{n_k} + 1$.

Suppose first that, at some stage, we end with $p_{k-1} = 5$. Then $p_{k-2} = 2 \cdot 5 + 1 = 11$, $p_{k-3} = 2 \cdot 11 + 1 = 23$, $p_{k-4} = 2 \cdot 23 + 1 = 47$, $p_{k-5} = 2 \cdot 47 + 1 = 95$. Since 95 is not a prime, $k = 1$, 2, 3, or 4. Hence

$$p_0 = 5, 11, 23, \text{ or } 47 \quad \text{and} \quad m = 5, 10, 11, 22, 23, 46, 47, \text{ or } 94.$$

Now, suppose that, at some stage, $p_{k-1} = 3$. Then $p_{k-2} = 2 \cdot 3^{n_{k-1}} + 1$ and $p_{k-3} = 2p_{k-2}^{n_{k-2}} + 1 = 2(2 \cdot 3^{n_{k-1}} + 1) + 1 = 4 \cdot 3^{n_{k-1}} + 3$ is not a prime. Hence $k = 1$ or 2, $p_0 = 3^{n_0}$ or $2 \cdot 3^{n_1} + 1$ and $m = 3^n$, $2 \cdot 3^n$, $2 \cdot 3^n + 1$, or $2(2 \cdot 3^n + 1)$ where $2 \cdot 3^n + 1$ is a prime.

Finally, we note that, if we do not come to 3 or 5, the process must continue indefinitely. But since $p_0, p_1, \cdots$, *etc.*, form a decreasing sequence of primes, this is impossible.

4. Conclusions. The results obtained above give us the

THEOREM. *The chain of automorphism groups* $\mathfrak{G}_0, \mathfrak{G}_1, \mathfrak{G}_2, \cdots$, *consists entirely of cyclic groups if and only if* $\mathfrak{G}_0$ *is cyclic of order* m *where* $m = 1, 2, 4, 5, 10, 11, 22, 23, 46, 47, 94, 3^n, 2 \cdot 3^n, 2 \cdot 3^n + 1$, *or* $2(2 \cdot 3^n + 1)$, *where* $2 \cdot 3^n + 1$ *is a prime.*

Because the only Abelian groups with Abelian automorphism groups are the cyclic groups* the theorem above may be restated with the requirement that $\mathfrak{G}_0, \mathfrak{G}_1, \cdots$ be Abelian.

* G. A. Miller, On the groups which have the same group of isomorphisms, Trans. Amer. Math. Soc., vol. 1, 1900, pp. 395–401. It is interesting to note that this fact does not seem to be at all well known. Thus Kurosh, in his recent book on group theory, simply mentions that an Abelian group *need* not have an Abelian automorphism group and cites one example.

A TOURNAMENT PROBLEM

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Introduction. In his book,† Steinhaus discusses the problem of ranking n objects according to some transitive characteristic, by means of successive pairwise comparisons. In this paper we shall adopt the terminology of a tennis tournament by n players. The problem may be briefly stated: "What is the smallest number of matches which will always suffice to rank all n players?"

Steinhaus proposes an inductive method whereby, the first k players having been ranked, the $(k+1)$-st player is matched against the median player in the first k, and by a "halving" process is finally ranked into this chain. Then the $(k+2)$-nd player is ranked into the new chain of $k+1$ players in the same manner.

Using this process, a player can be ranked into a chain of k others in $S(k) = 1 + [\log_2 k]$ matches. Steinhaus thus shows that $M(n)$ matches always suffice for n players where

$$M(n) = 1 + nS(n) - 2^{S(n)}.$$

He then states, "It has not been proved that there is no shorter proceeding possible, but we rather think it to be true."

The purpose of this note is to present an improved procedure, compare it with Steinhaus' $M(n)$ as an upper bound and with a lower bound $L(n)$ derived from information theory, and to discuss the asymptotic behavior of these three functions for large n.

A lower bound, $L(n)$, is easily seen to be $L(n) = 1 + [\log_2 (n!)]$, since each pairing can do no more than divide the remaining possibilities into two complementary sets; the results of the comparison then selects one or the other of these. Observing $n!$ possibilities initially, with halving the best we can do at each stage, we are led directly to the above formula for $L(n)$.

The improved procedure. This may be explained inductively in three steps (illustrated for the case $n = 19$ by Fig. 1). Suppose $n = 2r$ or $2r+1$.

1. Pair off $2r$ of the players and let the pairs play in the first round leaving one man out if n is odd.

2. By a continuation of the present method applied to r players, give a complete ranking of these r first round winners.

3. The third step is best explained by a diagram.

At this point in the ranking we have a hierarchy of the form illustrated in Figure 1 for $n = 19$. First round winners $J, I, \cdots, B$ are ranked in that order with J the best player. A is the first round loser to B, and other first round losers

* Now at General Analysis Corporation.

† H. Steinhaus, Mathematical Snapshots, New York, 1950, pp. 37–40.

are indicated directly below their respective victors. The odd man drawing a first round bye is considered a loser and put in the position at the extreme left of the diagram.

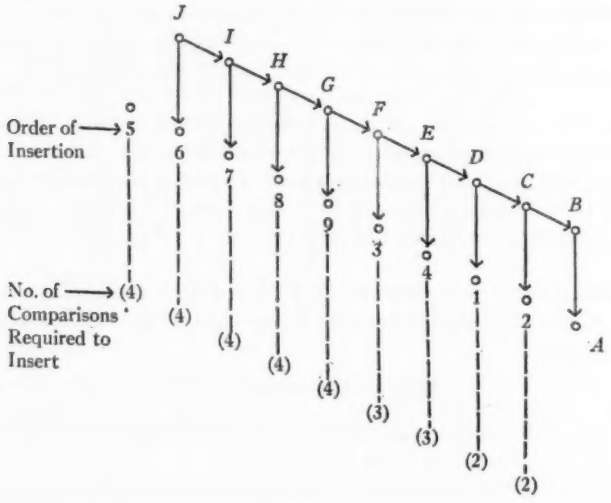

FIG. 1

The phrase "main chain" initially will refer to the chain $JIH \cdots CBA$, and the procedure will be to insert the numbered points in the main chain in the order indicated. The procedure is based on the fact that the insertion of a single point into a chain by the Steinhaus method is most efficient if the number of points on the chain is of the form 2^K-1.

Hence we start by inserting point 1 in the chain ABC. After this has been done, the "main chain" under point 2 consists of AB and possibly point 1; this insertion can also be performed with two comparisons.

We now turn to chains of length $2^3-1=7$, and observe that point 3 is as high as we can go, the main chain under 3 being composed of $ABCDE$ and 1 and 2, and so forth.

This represents a ranking technique, requiring $U(n)$ comparisons, where $U(n)$ is given recursively as follows.

$$U(1) = 0, \qquad U(2) = 1,$$

$$U(2k) = k + U(k) + \sum_{i=2}^{k} T(i),^*$$

$$U(2k+1) = k + U(k) + \sum_{i=2}^{k+1} T(i),$$

* The $T(i)$ are exactly the parenthetical numbers along the bottom of Figure 1.

where

$$T(i) = 2 \quad \text{for} \quad 1 < i \leqq 3,$$
$$T(i) = 3 \quad \text{for} \quad 3 < i \leqq 5,$$
$$T(i) = 4 \quad \text{for} \quad 5 < i \leqq 11,$$
$$\cdots\cdots\cdots\cdots$$
$$T(i) = j \quad \text{for} \quad t_{j-1} < i \leqq t_j,$$

and $t_j = \{2^{j+1} + (-1)^j\}/3$.

We shall now give a table of values of $M(n)$, $U(n)$, and $L(n)$ for some selected values of n, thereafter making some empirical observations.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$M(n)$	0	1	3	5	8	11	14	17	21	25	29	33	37
$U(n)$	0	1	3	5	7	10	13	16	19	22	26	30	34
$L(n)$	0	1*	3	5	7	10	13	16	19	22	26	29	33

FIG. 2

It may be observed that $U(n) = L(n)$ for $n = 20$ and 21 also; thus the proposed procedure is optimal for those values of n in addition to the values $n \leqq 11$. We conjecture that the value $L(12) = 29$ can not be achieved, but it seems to be difficult of proof. We further conjecture that $U(n)$ is best possible, for all n, but have no mathematical grounds on which to attack a similar conjecture regarding $L(n)$.

Asymptotic formulae. We state the following formulae for the asymptotic behavior of $U(n)$, $L(n)$, and $M(n)$. The proofs are sufficiently cumbersome to be omitted. (The formulae for $U(n)$ and $M(n)$ are based on the subsequence of n which appear to give local minima for $U(n) - L(n)$, *i.e.*, $n = [2^k/3]$ for some k. These being the "best" values, the comparison may be somewhat invidious.) The decimal values which appear as coefficients for n are, of course, approximations.

$$M(n) \sim n \log_2 n - .915n + O(\log_2 n),$$
$$U(n) \sim n \log_2 n - 1.415n + O(\log_2 n),$$
$$L(n) \sim n \log_2 n - 1.443n + O(\log_2 n).$$

On the other hand, for the subsequence of $n = 2^k$ for some k, which are the "best" values for $M(n)$, the coefficients of n in the expressions for $M(n)$ and $U(n)$ are -1 and -1.333, respectively.

* A more correct statement for $L(n)$ should be "the least integer $K \geqq \log_2 n!$." Our formula is equivalent except for $n = 2$.

DESCARTES AND THE GEOMETRIZATION OF ALGEBRA

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The great accomplishment of Descartes in mathematics invariably is described as the arithmetization of geometry [1]. Is it not true that his celebrated *Géométrie* of 1637 is far more concerned with algebra than with geometry [2]? Have we not read [3] that he "freed himself completely from the superstition of homogeneity?" Did not Descartes express the fundamental principle of analytic geometry?

> For the solution of any one of these problems of loci is nothing more than the finding of a point for whose complete determination one condition is wanting. . . . In every such case an equation can be obtained containing two unknown quantities [4].

At least one writer [5] consequently has jumped to the conclusion that "The real objects with which he dealt were numbers." It is the purpose of this paper to point out that all such assertions are but one side of a two-faced coin, for the truth is that Descartes had no intention of arithmetizing geometry. In fact, the purpose of *La géométrie* might with equal validity be described as the translation of algebraic operations into the language of geometry. The very first section of the work is entitled "How the calculations of arithmetic are related to the operations of geometry," and the second describes "How multiplication, division, and the extraction of square roots are performed geometrically." Also included in Book I of *La géométrie* are detailed instructions on the solution of quadratic equations, not in the algebraic sense of the ancient Babylonians or the medieval Arabs or the modern cossists, but geometrically, as in classical Greece. Book II of *La géométrie*, "On the nature of curved lines," comes close to the subject which now is known as analytic geometry, but Book III reverts to the theme of making algebra intelligible through geometric constructions of the roots of polynomial equations.

La géométrie was published as an appendix to the celebrated *Discours de la méthode*, and in the main treatise Descartes seems to have been partial neither to geometry nor to algebra. The former he accused of relying too heavily on diagrams which fatigue the imagination unnecessarily, and the latter he criticized as a confused and obscure art which embarrasses the mind [6]. The aim of his method, then, was two-fold: (1) Through algebraic procedure to free geometry from the use of diagrams, and (2) to give meaning to the operations of algebra through geometric interpretation. His work was indeed a linking of the two fields of geometry and algebra, but the association was bipolar and not prejudiced in favor of either direction.

The significance of Descartes' accomplishment becomes clearer when one recalls that the circumspect mathematicians of ancient Greece, shocked by the discovery of the incommensurable, had divided their field into two distinct halves—geometry (the study of continuous magnitude) and arithmetic (the

study of discrete number). An equation could form part of either half, but not both. If it arose in geometry, the unknown magnitude necessarily was to be exhibited as a line, an area, or a volume; in arithmetic the answer had to be either a whole number or a ratio of whole numbers. Quadratic equations, for example, appear both in Euclid's *Elements* and in Diophantus' *Arithmetic*, but the forms of the answers are quite different in the two cases. For cubic equations the geometric point of view predominated in Greece after it was found that such equations could be solved graphically through intersecting conic sections. The arithmetical equation $x^3=2$, for example, could not be solved, but the geometrical equation $x^3=2a^3$ was solved by finding the coordinate lines corresponding to the points of intersection of the parabolas $x^2=ay$ and $y^2=2ax$.

The decline of Greece and the rise of the Hindu and Arabic cultures led to the uncritical acceptance of irrational numbers, and the equation $x^3=2$ became arithmetically solvable. Until the early sixteenth century, however, it was generally believed that most cubic equations were arithmetically unsolvable, even in terms of radicals. The discovery, by del Ferro, Tartaglia, and Cardan, that all cubic equations are solvable numerically was a triumph which led to the development of algorithmic algebra without recourse to geometry, a direction which was encouraged by the discovery that imaginary cube roots could combine to give real numbers. Viète, the greatest algebraist of the sixteenth century, handled his equations so deftly by algebraic rules, despite his observance of the principle of homogeneity, that Descartes later criticized his work as marking too great a separation of algebra from geometry [7]. The obscurities of the new algebra repelled Descartes, and he set out to show that this subject, like geometry, is simply a description of magnitude. That is, Descartes in a sense was returning in thought to the ancient geometrical algebra, while at the same time he encouraged the development of symbolic forms of expression. In so doing Descartes initiated a great reform in mathematics, but he scarcely anticipated the form that this was to take in the nineteenth-century arithmetization of the subject.

Descartes was convinced that all mathematical sciences proceed from the same basic principles, and he decided to use the best of each branch. He began, in the *Discours*, by supposing all magnitudes, without exception, to be represented by lines, for he could think of nothing simpler which could be represented more distinctly to his imagination and his reason. But in order to hold them in mind and to understand them collectively, he found it necessary to express them in ciphers. In this way he felt that he would borrow the best of both geometric analysis and algebra, correcting the faults of each [8].

Descartes recognized that there are distinct advantages—in generality and in facility of handling—in thinking of every magnitude, whether an area, a volume, or any other quantity, as a line. It enables one to transcend three dimensions, and it obviates the necessity for a strict homogeneity in expressions to be combined. Descartes showed that one now could take the cube root of quantities such as a^2b^2-b, for one can think of the first term as divided once by

the unit line segment and the latter as multiplied twice by this unit. The cube root then will take the form which Descartes preferred; it will be a line segment [9]. Two things in particular should be noted here. In the first place, Descartes did not exactly abolish the principle of homogeneity; he simply substituted a mental geometric homogeneity for an apparent algebraic inhomogeneity. Secondly, and more importantly, it will be remarked that Descartes' basic magnitudes are not numbers at all; they are line segments, in conformity with the emphasis which Descartes placed on the idea of extension in his philosophical system. Descartes' approach would geometrize even arithmetic itself, oblivious of the qualms felt by Greek thinkers (qualms stilled much later by the Cantor-Dedekind axiom), and at least one seventeenth-century commentator [10] subsumed many of the Cartesian constructions under the caption "Arithmétique par Géométrie."

Descartes was not thoroughgoing in his geometrization of numbers, despite his realization that one of the chief advantages of algebra was its generality. Statements found in many a reputable history of mathematics notwithstanding, he did not systematically explain the graphical status of negative numbers [11], and imaginary numbers were completely beyond the geometric pale. In a vague sort of way Descartes realized that when a line turned out to be negative it was to be drawn in a sense contrary to that which had been taken as positive, but so little did he appreciate the significance of this that in 1638 he sketched his folium as a leaf only, restricting himself to positive values of the coordinates [12]. Occasionally he made use of negative ordinates, but not of negative abscissas.

The cavalier attitude of Descartes toward negative coordinates undoubtedly was a consequence of the goal of *La géométrie*. The object of the work was not the sketching of curves, as is now often assumed. There is in the whole of *La géométrie* not a single new curve plotted directly from its equation. Descartes' purpose, as brought out in the last book of the treatise, is the geometric construction of the roots of polynomial equations, using intersecting curves of lowest possible degree. This is, of course, reminiscent again of Greek geometric algebra, where the search for a geometric solution of $x^3 = 2a^3$ had led to the discovery of the conic sections. That so much of *La géométrie* is concerned with the theory of equations—the number of possible positive roots, increasing and decreasing the roots of an equation, finding rational roots, and depressing the degree when a root is known—does not indicate a preference for algebra over geometry. It stems from the need to know, if one is to use the simplest possible curves for the geometric solution, whether or not a given equation is reducible [13]. While Descartes did indeed convert geometric problems into the language of algebra, this was not an end in itself; it was but an intermediate step in determining the most appropriate geometrical construction. Consequently his theory of equations, being an algorithm of lines rather than of numbers, is not more appropriately described as an arithmetization of geometry than as a geometrization of algebra.

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NOMOGRAPHIC SOLUTIONS FOR POSITION RELATIONS BETWEEN A CLOSE EARTH SATELLITE AND ITS OBSERVER

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Introduction. The routine ephemeris predictions for an artificial satellite can be expected to give for any instant only the latitude and longitude of the subsatellite point, together with the height of the satellite above the earth's surface. The problem would then remain to determine, for an individual observer not directly under the satellite, its apparent azimuth, and altitude or elevation above the horizon; also its distance or slant range.

Although the formulas for such relations are quite simple and easy to apply for those versed in computation, for repeated use requiring only moderate accuracy, and especially for nontechnical observers not possessing mathematical tables, simple charts or nomograms for quick graphical solutions are desirable.

Solution for the ground range and azimuth. The ordinary astronomical triangle of spherical trigonometry yields the following equation for the arc distance or ground range $d°$ of the subsatellite point as related to the observer's

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latitude ϕ_0, the satellite latitude ϕ, and the longitude difference $\Delta\lambda$:

$$\cos d^\circ = \sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos \Delta\lambda. \tag{1}$$

The law of sines gives the satellite's apparent azimuth A:

$$\sin A = \sin \Delta\lambda \cos \phi \csc d^\circ. \tag{2}$$

Numerical solutions for these equations are tabulated in *Hydrographic Office Publication No. 214.* A universal graphical solution has been published in the form of a single nomogram by Hughes [1]. It consists of two parallel scales and a network of curved line scales, one curve for every 5° of observer's latitude. A more workable chart would be obtained by omitting all but two adjacent curved line scales, printing up all such combinations and choosing for each observer the chart appropriate to his latitude interval. Another simplification would be to print a separate chart for the satellite latitude—observer latitude—longitude difference—ground range relation, and another for the satellite latitude—longitude difference—ground range—azimuth relation. In the published tables and charts for which notations fit the astronomical triangle, replace *Hour Angle* by *Longitude Difference*, *Declination* by *Satellite Latitude*, and *Altitude* by 90° minus *Range*.

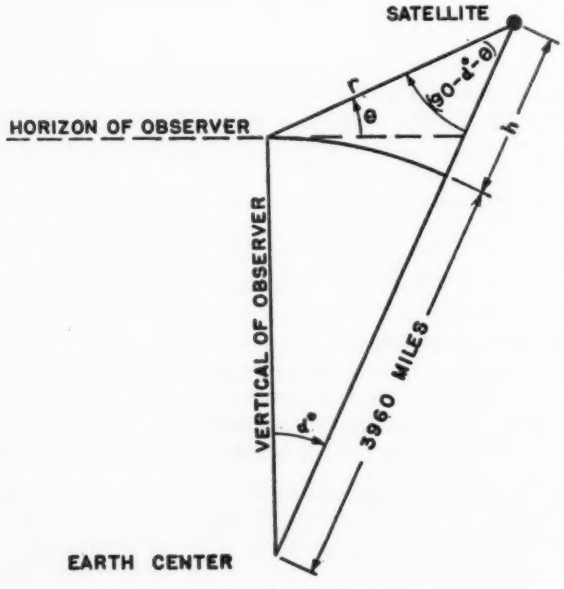

FIG. 1

Solution for the apparent altitude or elevation. The elevation θ is related to the ground range $d°$ and the satellite height h in miles (Fig. 1) by the equation

$$\tan\theta = \cot d° - \left(\frac{3960}{3960+h}\right)\csc d°. \tag{3}$$

This equation, like (1), is of the form [2]

$$f_1(u) = f_4(w) - f_2(v)f_3(w) \tag{4}$$

depictable on a nomogram consisting of two parallel scales and a curved scale. Two parallel vertical scales, linear in $\tan\theta$ and $3960/(3960+h)$ (but labelled only in θ and h) are constructed with an arbitrary metric separation of K. Then, if m_θ and m_h are the metric lengths of units of $\tan\theta$ and $3960/(3960+h)$, respectively, also if the X-axis is horizontal through the zero points of these functions, with origin at the left scale, the metric coordinates of points on a curved $d°$ scale are given by

$$X_{d°} = \frac{Km_\theta \csc d°}{m_\theta \csc d° + m_h}, \qquad Y_{d°} = \frac{m_\theta m_h \cot d°}{m_\theta \csc d° + m_h}. \tag{5}$$

A straightedge laid over the chart will then always intersect readings of θ, $d°$, and h which satisfy (3).

Solution for the distance or slant range. The slant range r in miles is given by

$$r = (3960 \sin d°)[\sec(d° + \theta)] \tag{6}$$

which may be represented by a so-called Z-chart, an alignment chart constructed as follows [3]. Parallel vertical scales of arbitrary separation, linear in r and $3960 \sin d°$ (but labelled only in r and $d°$), have the first with zero point at the bottom and the second with zero point at the top, and a diagonal of metric length K connecting these zero points. Then, if m_r and $m_{d°}$ are the metric lengths of units of r and $3960 \sin d°$, respectively, the distance along the diagonal from its upper right end to a point corresponding to $d°+\theta$ is given by

$$X_{d°+\theta} = \frac{K}{(m_r/m_{d°})\sec(d° + \theta) + 1}. \tag{7}$$

A straightedge will then always intersect values of r, $d°+\theta$, and $d°$ which satisfy (6).

Summary. At each station for observing a close satellite there will be required only four simple nomograms and a straightedge to obtain from a routine published satellite ephemeris the satellite's azimuth, elevation, and slant range at any instant. The first two nomograms will give the azimuth and ground range, respectively, of the subsatellite point from the known geographical positions of observer and satellite, the third nomogram will give the apparent elevation of the satellite from its known ground range and linear height above the

earth, and the fourth nomogram will give the distance or slant range of the satellite from its known ground range and altitude. No computation or tables are required for such results, which will have accuracies of the order of 1° and 5 to 10 miles, depending only on the precision of scale readings at the intersections with an overlaid straightedge.

The author is indebted to Drs. John P. Hagen and Joseph W. Siry for discussion of this problem, and to Mrs. Mary P. Hann and Miss Ann Eckels for computing and drafting the nomogram scales.

Example. Suppose an observer, located at latitude 35° N and longitude 85° W, is looking for a satellite at latitude 30° N, longitude 70° W, and height 300 miles. It is required to find the satellite's azimuth, elevation, and slant range.

1. On the NOMOGRAM FOR SATELLITE GROUND RANGE, LATITUDES OF OBSERVER AND SATELLITE, AND LONGITUDE DIFFERENCE, lay a straightedge between the 15° point on the scale for Satellite-Observer Difference in Longitude and the intersection of the area marked "North and North" of curves marked "Satellite Latitude 30°" and "Observer's Latitude 35°." (For convenience, note that the base of the latter curve on the "Ground Range" scale is at $90° - 35° = 55°$.) The straightedge will then intersect the Ground Range scale at a point which may be estimated as about 13°. This ground range is in degrees of arc on a great circle; for present purposes there is no need to convert it to statute or nautical miles.

2. In examples such as the present one, where the difference in longitude is less than 20°, ground range may be more conveniently and accurately determined on the APPROXIMATE NOMOGRAM FOR GROUND RANGE, AND LATITUDE AND LONGITUDE DIFFERENCES BETWEEN OBSERVER AND SATELLITE. On this lay one straightedge between the 15° point on the $\Delta\lambda$ scale and $\phi_0 = 35°$ on the diagonal scale above it. From the point where this intersects the horizontal T-scale, extend a second straightedge to the 5° point on the $\Delta\lambda$ scale at the top. The second straightedge is then seen to intersect the $d°$ (ground range) scale at 13.6°.

3. On the NOMOGRAM FOR SATELLITE AZIMUTH, GROUND RANGE, AND LATITUDES OF OBSERVER AND SATELLITE, extend a straightedge between the 30° point on the Satellite Latitude scale and the intersection of curves marked "Observer's Latitude 35°" and "Ground Range—Observer North" corresponding to 13.6°. (For convenience note that the latter curve would intersect the Observer's Latitude scale at $90° - 13.6° = 76°$.) The straightedge will then intersect the Azimuth from North scale at 107°.

4. On the NOMOGRAM FOR HEIGHT, GROUND RANGE, AND ELEVATION ANGLE, lay a straightedge between the 300-mile point on the h (height) scale at the extreme right and the point on the $d°$ (ground range) curve to its left corresponding to $d° = 13.6°$. This straight edge will then intersect the θ (elevation) scale at 10.2°, the apparent altitude of the satellite above the horizon.

5. On the NOMOGRAM FOR GROUND RANGE, SLANT RANGE, AND ELEVATION ANGLE, lay a straightedge between the 13.6° point on the $d°$(ground range) scale at the right and the point on the diagonal scale corresponding to $d° + \theta = 13.6° + 10.2° = 23.8°$. This straightedge will then intersect the r (slant range) scale at 1018 statute miles.

Larger scale charts are available on request to Code 4143, Beltsville Space Center, National Aeronautics and Space Administration, Washington 25, D. C.

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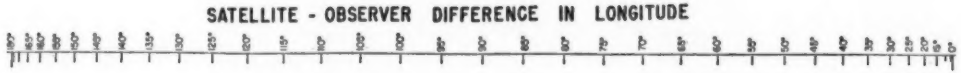
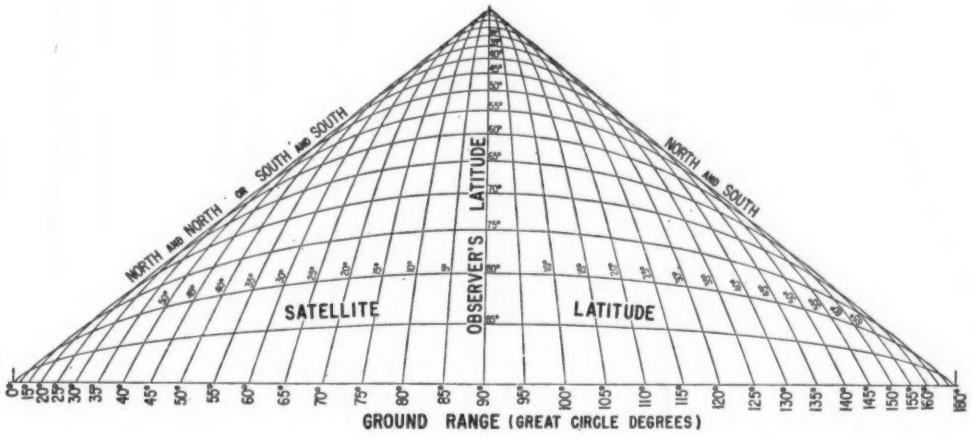


NOMOGRAM FOR SATELLITE GROUND RANGE, LATITUDES OF OBSERVER AND SATELLITE AND LONGITUDE DIFFERENCE

(Straight Edge Connects Corresponding Values Using the Intersection of Satellite and Observer Latitudes)



APPROXIMATE NOMOGRAM FOR GROUND RANGE, AND LATITUDE AND LONGITUDE DIFFERENCES BETWEEN OBSERVER AND SATELLITE

(Two Straight Edges Intersecting on T-Scale Connect Corresponding Values)

$(d^\circ)^2 = (\Delta\phi)^2 + (\Delta\lambda)^2 \cos^2 \phi_0$

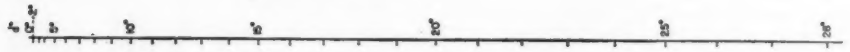
useful approximation when $\Delta\lambda < 20^\circ$ and $\phi_0 < 60^\circ$

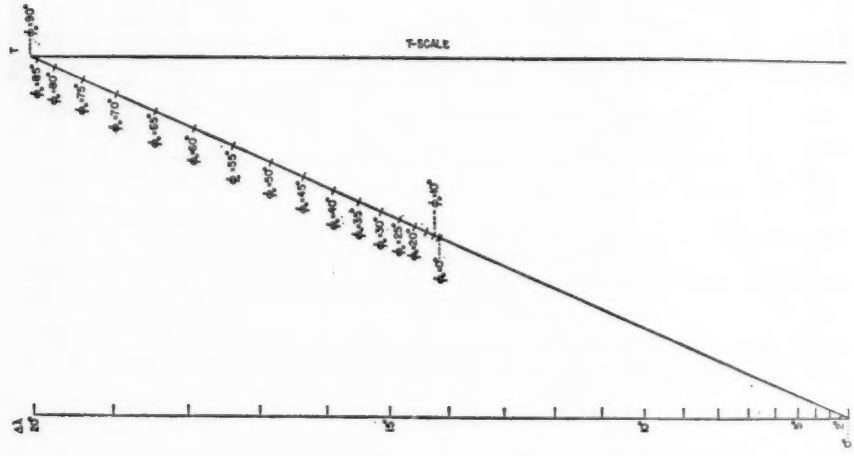
d° = ground range

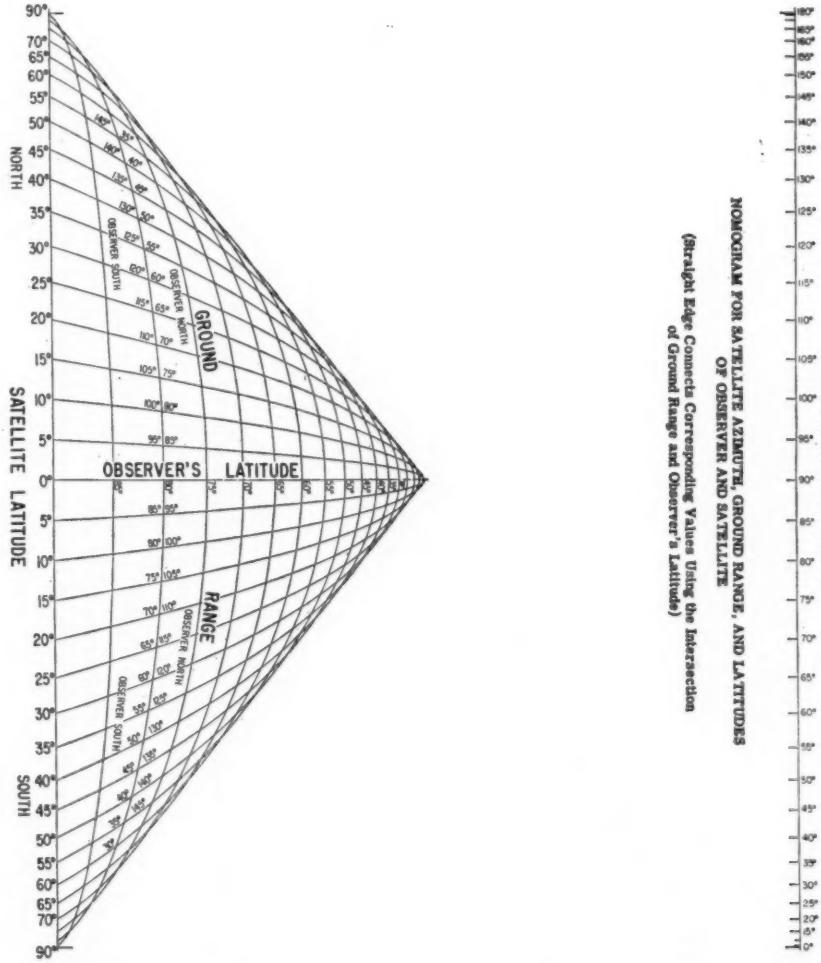
$\Delta\phi$ = difference in latitude between observer and satellite

$\Delta\lambda$ = difference in longitude between observer and satellite

ϕ_0 = latitude of observer

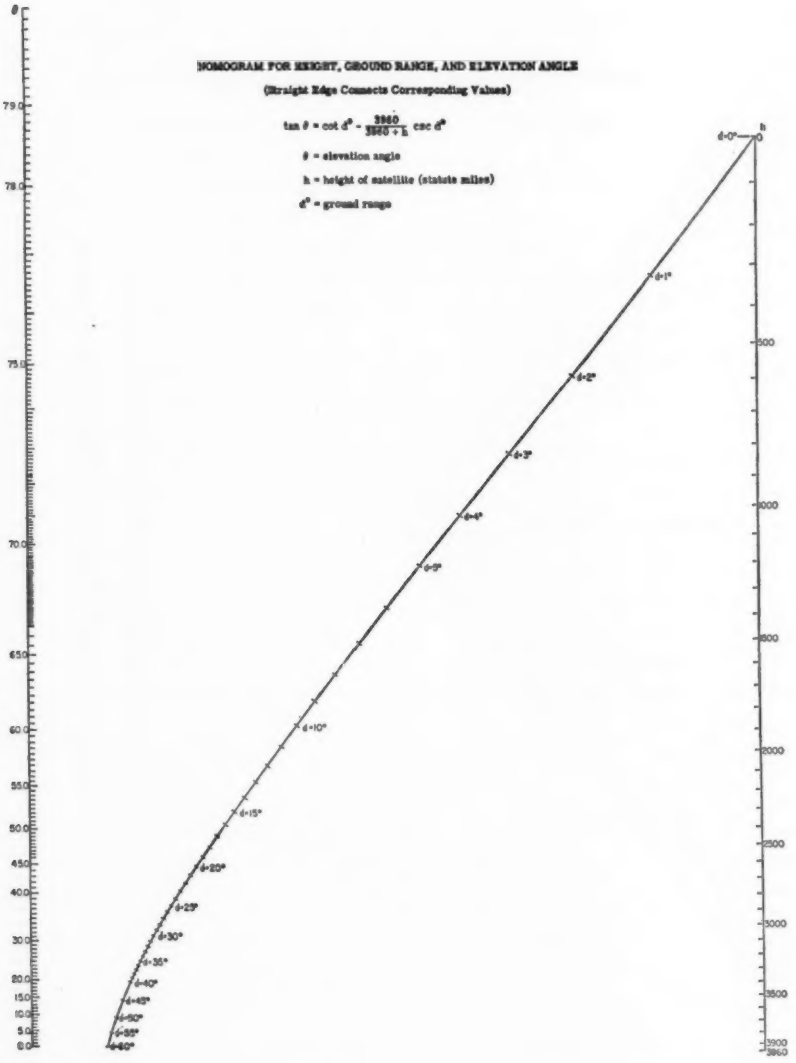


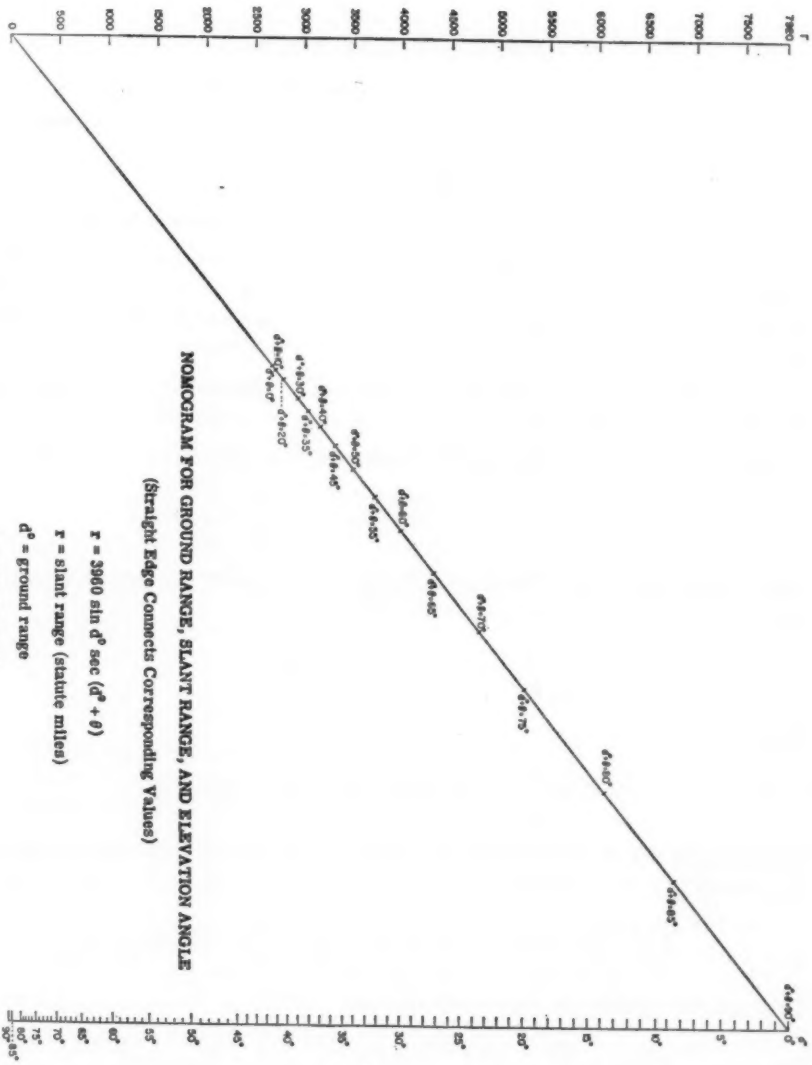




NOMOGRAM FOR SATELLITE AZIMUTH, GROUND RANGE, AND LATITUDES OF OBSERVER AND SATELLITE

(Straight Edge Connects Corresponding Values Using the Intersection of Ground Range and Observer's Latitude)


θ
79.0
78.0
75.0
70.0
65.0
60.0
55.0
50.0
45.0
40.0
30.0
20.0
15.0
10.0
5.0
0.0
HOMOGRAM FOR HEIGHT, GROUND RANGE, AND ELEVATION ANGLE
(Straight Edge Connects Corresponding Values)
tan θ = cot d° - 3960/(3960 + h) csc d°
θ = elevation angle
h = height of satellite (statute miles)
d° = ground range
d=0°
h
0
d=1°
500
d=2°
1000
d=3°
d=4°
d=5°
1500
d=10°
2000
d=15°
2500
d=20°
d=25°
3000
d=30°
d=35°
3500
d=40°
d=45°
d=50°
d=55°
3900
d=60°
3960



NOMOGRAM FOR GROUND RANGE, SLANT RANGE, AND ELEVATION ANGLE

(Straight Edge Connects Corresponding Values)

$r = 3960 \sin d^\circ \sec (d^\circ + \theta)$

r = slant range (statute miles)

d° = ground range

θ = elevation angle

MATHEMATICAL NOTES

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A RODRIGUES' FORMULA

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In this note we shall discuss a set of polynomials which is defined by a generating function and is shown to satisfy a Rodrigues' formula and several recurrence relations. A portion of this work generalizes the results of a paper by Humbert [1] concerning a set of polynomials of even degree.

DEFINITION. *A k-set of polynomials* $\{P_{kn}(x)\}$ *is a sequence of polynomials such that* $P_{kn}(x)$ *is of exactly degree* kn, $n=0, 1, 2, \cdots$; k *a natural number.*

The k-set with which we shall be concerned is generated by

$$g(x, t) = (1 - t)^{-1} \exp [x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}(x)t^n,$$

where $u(t)=1-(1-t)^{-k}$. The k-set $\{T_{kn}(x)\}$ is determined by a recurrence relation which we shall proceed to develop. We see that

$$\frac{\partial}{\partial x} g(x, t) = g(x, t)[kx^{k-1}u(t)].$$

This may be written as

$$(1 - t)^k \frac{\partial}{\partial x} g(x, t) = kx^{k-1}[(1 - t)^k - 1]g(x, t).$$

Writing each side of this equation as a power series in t and equating corresponding powers of t, we find that

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^i T'_{k(n-i)}(x) = kx^{k-1} \sum_{i=1}^{k} \binom{k}{i} (-1)^i T_{k(n-i)}(x).$$

This equation yields the recurrence relation

$$T'_{kn}(x) = \sum_{i=1}^{k} (-1)^i \binom{k}{i} [kx^{k-1}T_{k(n-i)}(x) - T'_{k(n-i)}(x)]. \tag{1}$$

This recurrence relation may be used to show that the k-set $\{T_{kn}(x)\}$ satisfies a Rodriques' formula. However another method is more convenient.

Maclaurin's theorem allows us to write

$$g(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n} g(x, 0)\right] t^n = \sum_{n=0}^{\infty} T_{kn}(x) t^n,$$

so that

$$T_{kn}(x) = \frac{1}{n!} \frac{\partial^n}{\partial t^n} g(x, 0)$$

$$= \frac{\exp(x^k)}{n!} \frac{\partial^n}{\partial t^n} [(1-t)^{-1} \exp[-x/(1-t)]^k]_{t=0} \tag{2}$$

$$= \frac{\exp(x^k)}{n!} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^{kr}(kr+1)_n,$$

where $(kr+1)_n = (kr+1)(kr+2)\cdots(kr+n)$. On the other hand we find

$$D^n[x^n \exp(-x^k)] = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^{kr}(kr+1)_n.$$

Comparing this with the previous result we are able to state

THEOREM 1. *The k-set* $\{T_{kn}(x)\}$ *defined by the generating function*

$$g(x, t) = (1-t)^{-1} \exp(x^k[1-(1-t)^{-k}])$$

satisfies the Rodrigues' formula $T_{kn}(x) = \exp(x^k) D^n[x^n \exp(-x^k)]/n!$.

The explicit formula

$$T_{kn}(x) = \frac{1}{n!} \sum_{p=0}^{n} \frac{x^{kp}}{p!} \sum_{r=0}^{p} (-1)^r \binom{p}{r} (kr+1)_n$$

is a direct consequence of equation (2).

As an application of Theorem 1 we show that the k-set $\{T_{kn}(x)\}$ satisfies a property related to orthogonality.

THEOREM 2. *The moments of* $\{T_{kn}(x)\}$ *with respect to the weight function* $w(x) = \exp[-x^k]$ *have values*

$$M(s, n) = \begin{cases} 0, & s = 0, 1, \cdots, n-1; \\ (-1)^n \binom{s}{n} \frac{n!}{k} \Gamma\left(\frac{s+1}{k}\right), & s = n, n+1, \cdots, \end{cases}$$

where $M(s, n) = \int_0^{\infty} w(x) x^s T_{kn}(x) dx$.

Proof. By means of the Rodrigues' formula we have

$$M(s, n) = \int_0^{\infty} x^s D^n[x^n w(x)] dx$$

$$= \sum_{i=0}^{r-1} (-1)^i i! x^{s-i} D^{n-1-i}[x^n w(x)] + (-1)^r \binom{s}{r} r! \int_0^\infty x^{s-r} D^{n-r}[x^n w(x)]dx.$$

This expression is the result of the rth integration by parts.

If $s=0, 1, \cdots, n-1$ then for $r=s$ we obtain $M(s, n)=Q(x)w(x)\,]_0^\infty=0$, since $Q(x)$ is a polynomial with the property $Q(0)=0$.

If $s=n, n+1, \cdots$ then for $r=n$ we have

$$M(s, n) = (-1)^n \binom{s}{n} n! \int_0^\infty x^s w(x)dx = (-1)^n \binom{s}{n} \frac{n!}{k} \Gamma\left(\frac{s+1}{k}\right).$$

For the case $k=1$ the polynomial set is the Laguerre type of order zero and the property of Theorem 2 is equivalent to orthogonality.

The Rodrigues' formula may also be used to develop a particularly simple recurrence relation. We rewrite the formula as

$$(r+1)! \exp(-x^k) T_{k(r+1)}(x) = D^{r+1}[x \cdot x^r \exp(-x^k)]$$

and apply the Leibniz formula. After some rearrangement we obtain the recurrence relation

$$(r+1)T_{k(r+1)}(x) = xT'_{kr}(x) + (r+1-kx^k)T_{kr}(x). \tag{3}$$

As a closing remark we mention that the k-set $\{T_{kn}(x)\}$ satisfies a differential equation of order $k+1$. We make use of the relation (1) rewritten as

$$T'_{k(n+k)}(x) = \sum_{i=1}^{k} (-1)^i \binom{k}{i} [kx^{k-1} T_{k(n+k-i)}(x) - T'_{k(n+k-i)}(x)] \tag{4}$$

and

$$(r+1)T^{(m)}_{k(r+1)}(x) = xT^{(m+1)}_{kr}(x) + (r+m+1-kx^k)T^{(m)}_{kr}(x)$$

$$- k \sum_{i=1}^{m} \binom{m}{i} \binom{k}{i} i! x^{k-i} T^{(m-i)}_{kr}(x), \qquad m \leqq k, \tag{5}$$

which is an immediate consequence of relation (3). An iterative procedure with the relations (3) and (5) yields

$$(r+1)_m T_{k(r+m)}(x) = x^m T^{(m)}_{kr}(x) + \sum_{i=0}^{m-1} p_{im}(x) T^{(i)}_{kr}(x), \tag{6}$$

where $p_{im}(x)$ is a polynomial of degree $ik-i+m$. When relation (6) and its derivative are inserted into (4) it follows that $T_{kn}(x)$ satisfies a homogeneous linear differential equation of order $k+1$ with polynomial coefficients.

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AN ELEMENTARY THEOREM ON GRAPHS

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The following theorem, while very easy to state and prove, does not appear to be in the literature. (See König [1] as a general reference on graph theory.) Let G be a connected graph and r be the number of cut points of G. A *block* of a graph is a maximal connected subgraph containing no cut points of itself. Let N be the number of blocks of G. Let n_i be the number of components of the subgraph of G obtained on removing the ith point, and let the points of G be numbered so that the cut points are the first r points. Then,

$$N - \sum_{i=1}^{r} n_i + r = 1. \tag{1}$$

Before proving this result, we verify it for trees, *i.e.*, connected graphs with no cycles. For any graph G, let p and q be the number of points and the number of lines respectively. If G is connected and m is the number of independent cycles of G, then the well-known Euler characteristic can be stated in the form

$$p - q + m = 1. \tag{2}$$

For trees, this specializes at once to

$$p - q = 1. \tag{3}$$

It is convenient in verifying equation (1) for trees to first restate it in the following form:

$$N - \sum_{i=1}^{p} n_i + p = 1. \tag{1'}$$

It is of course obvious that (1′) is equivalent to (1) since the quantity $p-r$ has been both added and subtracted to the left side of equation (1) to obtain (1′). We note that $n_i=1$ whenever $i>r$.

Now for trees, n_i is the degree of the ith point and $N=q$ since each line is a block. But the sum of the degrees of all the points of any graph is $2q$. Therefore for trees, equation (1′) immediately becomes equation (3).

We now prove (1) for arbitrary connected graphs by induction on r. If $r=0$, then $N=1$ and equation (1) is trivial. For connected graphs in which $r>0$, an *endblock* is a block of G which contains exactly one cut point of G. Just as every tree has at least one endpoint, so does every connected graph with more than one block contain at least one endblock. As our inductive hypothesis, we assume that the theorem is true for all connected graphs containing exactly r cut points and consider a connected graph G having exactly $r+1$ cut points. Clearly, when $r>1$ there exists a cut point c of G such that exactly one block at c is not an endblock. Let H be the graph obtained from G by removing all the end-

blocks of G which contain c. Then H has exactly r cut points and is connected. Therefore, by the inductive hypothesis, equation (1) holds for H. Thus if $N(H)$ is the number of blocks of H, then $N(H)-\sum_{i=1}^{r} n_i+r=1$. But obviously $N(G)-N(H)=n_{r+1}-1$. Therefore equation (1) holds for G and the proof is completed. We illustrate with Figure 1.

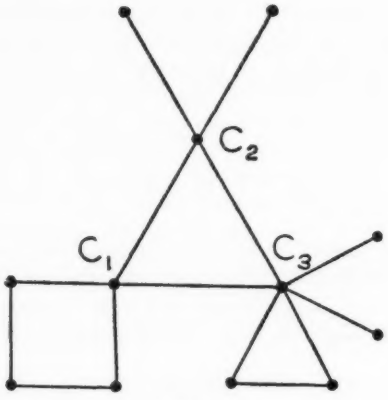

FIG. 1

Here $r=3$, $N=7$, $n_1=2$, $n_2=3$, $n_3=4$, and equation (1) is verified.

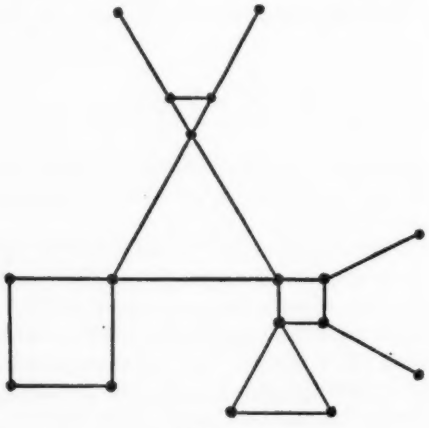

FIG. 2

There is a simple alternative proof of equation (1) which we now outline; it resembles one of the steps in a proof of the five color theorem. First, note that equation (1) is easily verified for connected graphs in which each cut point lies

on exactly two blocks. Then to prove equation (1) for any connected graph G, replace each cut point lying on more than two blocks by a cycle each of whose points is on exactly one of the blocks on which this point lies. We illustrate by showing in Figure 2 the graph obtained by modifying Figure 1 in this way. It is clear that such a modification of a connected graph does not affect the left-hand member of equation (1).

Reference

1. D. König, Theorie der endlichen und unendlichen Graphen. Leipzig, 1936; reprinted New York, 1950.

SOME FORMULAS PERTAINING TO A TETRAHEDRON*

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Bretschneider gave, without proof, the relations

$$a^2 + a'^2 + 2aa' \cot a \cot a' = b^2 + b'^2 + 2bb' \cot b \cot b' = c^2 + c'^2 + 2cc' \cot c \cot c' \tag{1}$$

among the lengths of the edges and the cotangents of the dihedral angles of a tetrahedron [1]. J. Hecquet rediscovered them and established them by elementary geometry [2]. R. Deaux and V. Thébault gave further details [3]. It might be of interest to deduce equations (1) directly from some formulas obtained by G. Dostor [4] and T. C. Lewis [5], as this gives an interpretation of the constant value of the quantities (1).

Let a, a', b, b', c, c' denote the lengths of the edges BC, DA, CA, DB, AB, DC, as well as the corresponding dihedral angles, of a tetrahedron $T \equiv ABCD$, and let (A), (B), (C), (D), V, and R denote the areas of the faces BCD, CDA, DAB, ABC of T, the volume of T, and the radius of the sphere $ABCD$. Let us set

$$a^2 + a'^2 = \alpha, \qquad b^2 + b'^2 = \beta, \qquad c^2 + c'^2 = \gamma.$$

According to Dostor [6], the volume of a tetrahedron is equivalent to two thirds of the product of two faces by the sine of the included dihedral angle divided by the length of the edge of this dihedral angle. Thus

$$V = 2(A)(D)(\sin a)/3a = 2(B)(C)(\sin a')/3a' = \cdots,$$

whence

$$V^2 = 4(A)(B)(C)(D)(\sin a)(\sin a')/9aa' = 4(A)(B)(C)(D)(\sin b)(\sin b')/9bb' = \cdots. \tag{2}$$

Furthermore, T. C. Lewis gives the formulas [5]

* Translated from the French by W. E. Byrne, Virginia Military Institute.

$$
\begin{aligned}
256(A)(B)(C)(D)\cos a\cos a' &= -288\alpha V^2 + H,\\
256(A)(B)(C)(D)\cos b\cos b' &= -288\beta V^2 + H,\\
256(A)(B)(C)(D)\cos c\cos c' &= -288\gamma V^2 + H,
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
H = {} & 144(\alpha+\beta+\gamma-4R^2)V^2\\
& + a^2a'^2(\alpha-\beta)(\alpha-\gamma)\\
& + b^2b'^2(\beta-\alpha)(\beta-\gamma)\\
& + c^2c'^2(\gamma-\alpha)(\gamma-\beta).
\end{aligned}
$$

After dividing any one of equations (3) by $288V^2$ and making use of (2) we obtain for the common value of the quantities (1),

$$
H/288V^2 = (\alpha+\beta+\gamma-4R^2)/2 + \sum a^2a'^2(\alpha-\beta)(\alpha-\gamma)/288V^2. \tag{4}
$$

If T is isosceles, relations (1) and (4) yield

$$
\begin{aligned}
2a^2/\sin^2 a = 2b^2/\sin^2 b = 2c^2/\sin^2 c\\
= \frac{3(a^2+b^2+c^2)}{4} + \frac{a^4(a^2-b^2)(a^2-c^2)+b^4(b^2-c^2)(b^2-a^2)+c^4(c^2-a^2)(c^2-b^2)}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.
\end{aligned}
$$

Because of (2), relations (1) also reduce to

$$
\alpha + k\cos^2 a = \beta + k\cos^2 b = \gamma + k\cos^2 c, \qquad k = 2S^4/9V^2,
$$

where S is the area of a face of T. It can also be shown that k is the ratio of the area of any face of T to the sine of the supplementary trihedral angle of the trihedral angle opposite this face.

If T is orthocentric, then $\alpha=\beta=\gamma$, and relations (1) reduce to

$$
\cos a\cos a' = \cos b\cos b' = \cos c\cos c',
$$

which characterizes this particular type of tetrahedron.

References

1. Archiv de Grunert, 1841.
2. Mathesis, 1955, p. 414; 1956, p. 487. Question 3713.
3. *Ibid.*
4. Archiv der Mathematik und Physik, t. 87, 1875, pp. 113–190.
5. The dihedral angles of a tetrahedron, Messenger of Mathematics, vol. 50, 1921, pp. 190–192.
6. Archiv der Mathematik und Physik, t. 58, p. 150.

CLASSROOM NOTES

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PARAMETER VARIATION AND THE SOLUTION OF BERNOULLI'S EQUATION

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Mr. A. N. Aheart in his note (this MONTHLY, vol. 58, 1951, pp. 696–697) has shown that the solution of the Bernoulli differential equation:

$$\frac{dy}{dx} + Py = Qy^n, \qquad n \neq 0, n \neq 1,$$

where P and Q are either constants or functions of x only, can be found directly without reducing it to the linear equation ($n=0$) by a proper substitution.

This note is to show that his method is similar to the method of variation of parameters used in determining the solution of the linear equation ([1], p. 20). In the following method, we can allow n to have the value zero, but not 1, in which case the variables can be separated and the equation integrated.

Consider the Bernoulli equation

$$\frac{dy}{dx} + Py = Qy^n, \qquad n \neq 1, \tag{1}$$

where P and Q are either constants or functions of x only.

The equation (1) with right-hand member equal to 0, gives after separating the variables and integrating:

$$y = ze^{-\int P dx}. \tag{2}$$

(If we set $u = e^{-\int P dx}$, (2) is the solution assumed by Mr. A. N. Aheart.)

Considering z as a parameter and differentiating with respect to x:

$$\frac{dy}{dx} = e^{-\int P dx}\left(\frac{dz}{dx} - Pz\right). \tag{3}$$

Substitution in (1) from (2) and (3) gives:

$$\frac{dz}{dx} = z^n Q e^{(1-n)\int P dx}.$$

Separating the variables and integrating:

$$\frac{z^{(1-n)}}{(1-n)} = C + \int Q e^{(1-n)\int P dx} dx,$$

where C is an arbitrary constant of integration.

Substitution in (2) gives the solution of (1) as:

$$y^{(1-n)}e^{(1-n)\int P\,dx} = (1-n)C + (1-n)\int Qe^{(1-n)\int P\,dx}dx.$$

[*Remark.* For $n=0$, this gives the solution of the linear equation $dy/dx+Py=Q$.]

Reference

1. A. R. Forsyth, A Treatise on Differential Equations, London, 1929.

THE INSOLVABILITY OF THE QUINTIC RE-EXAMINED

MORTON J. HELLMAN, Rutgers University

In the April 1958 issue of this MONTHLY a unifying technique for the solution of the quadratic, cubic, and quartic was presented using the relations between the roots and coefficients. It is interesting to pursue the editor's suggestion to see where this method breaks down in the case of the quintic. That is the purpose of this note.

It is sufficient to consider the reduced quintic

$$x^5 + ax^3 + bx^2 + cx + d = 0. \tag{1}$$

Denote the five roots by r_1, r_2, r_3, r_4, r_5 and set

$$r_1 + r_2 + r_3 = s, \qquad r_1r_2 + r_1r_3 + r_2r_3 = t,$$
$$r_1r_2r_3 = v, \qquad r_4 + r_5 = -s.$$
$$r_4r_5 = w,$$

Then the remaining four equations connecting the roots and coefficients are

$$-s^2 + t + w = a, \tag{2}$$

$$-ts + sw + v = -b, \tag{3}$$

$$tw - vs = c, \tag{4}$$

$$vw = -d. \tag{5}$$

From (5) $v=-d/w$ and from (4) $t=(c+vs)/w=(wc-ds)/w^2$ and substituting these relations in (2) and (3) we obtain, after simplification,

$$-w^2s^2 + wc - sd + w^3 = aw^2,$$
$$s^2d - wsc + sw^3 - dw = -bw^2$$

which upon elimination of either w or s yields an equation of higher degree than the fifth. Hence, as expected, this technique fails in the case of the quintic. It is easy to show, as one might expect, that equations (2) to (5) are equivalent to the equations which result when one attempts to factor (1) into cubic and quadratic factors.

COMPLEX ROOTS OF REAL POLYNOMIALS

J. P. BALLANTINE, University of Washington

1. Introduction. When a complex root, of the form $x+iy$, of a real polynomial, $P(z)$, is found by Newton's method, it is necessary to compute both $P(x+iy)$ and $P'(x+iy)$ for several trial values of $x+iy$. Substitution by the usual methods is laborious, since each complex multiplication entails four real multiplications.

The amount of calculation at each step is reduced by the Lin-Bairstow method* by avoiding almost all work with complex numbers. Instead of closing down on a complex root, the method approximates a real quadratic factor, which then can be solved for its complex roots.

The substitution of $x+iy$ into $P(z)$ and $P'(z)$ can be carried out in a manner, as we shall show, which avoids almost all work with complex numbers. With this method of substitution, Newton's method entails about the same amount of work as the Lin-Bairstow Method.

2. Quadratic synthetic division. Use is made of quadratic synthetic division of $P(z)$ by z^2-pz-q. The procedure is similar to ordinary synthetic division of $P(x)$ by $x-r$. A parameter, λ, has been introduced the purpose of which will appear later.

(1)

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 \cdots & a_{n-1} & a_n(p+q \\ & pb_0 & pb_1 & pb_2 \cdots & pb_{n-2} & \lambda pb_{n-1} \\ & & qb_0 & qb_1 \cdots & qb_{n-3} & qb_{n-2} \\ \hline b_0 & b_1 & b_2 & b_3 \cdots & b_{n-1} & b_n \end{array}$$

The value of each b_j under the line is found as the sum of the numbers in the same column above the line. That is:

$$b_j = a_j + pb_{j-1} + qb_{j-2}, \qquad 0 \leqq j < n, \tag{2}$$

$$b_n = a_n + \lambda pb_{n-1} + qb_{n-2}, \tag{3}$$

where it is assumed that all a_j and b_j are zero for j negative.

The quotient, $Q(z)$, and remainder, $R(z)$, of the division shown in (1) may be read as follows:

$$Q(z) = b_0z^{n-2} + b_1z^{n-3} + \cdots + b_{n-2}, \tag{4}$$

$$R(z) = b_{n-1}(z - \lambda p) + b_n. \tag{5}$$

The choice of λ is arbitrary. $R(z)$ does not depend on λ, since the λ appearing in (5) cancels with the λ in b_n. The usual choice is to take $\lambda=0$, and so omit all terms mentioned in this note which involve λ. The more useful choice, $\lambda=\frac{1}{2}$, will be explained later.

* Kaiser S. Kunz, Numerical Analysis, New York, 1957, pp. 34–37; W. E. Milne, Numerical Analysis, Princeton, 1949, pp. 53–57.

3. The substitution. First form a real quadratic having the factor $z-(x+iy)$, namely,

$$z^2 - pz - q = (z - x - iy)(z - x + iy)$$
$$= z^2 - 2xz + x^2 + y^2,$$

whence

$$p = 2x, \qquad q = -x^2 - y^2. \tag{6}$$

Now, divide $P(z)$ by z^2-pz-q, leaving a quotient $Q(z)$ and a remainder $R(z)$.

$$\begin{aligned} P(z) &= Q(z)(z^2 - pz - q) + R(z), \\ P(x + iy) &= 0 + R(x + iy), \\ &= b_{n-1}(x + iy - \lambda p) + b_n, \text{ by (5)}, \\ &= b_n + b_{n-1}(iy), \end{aligned} \tag{7}$$

by (6), provided $\lambda=\frac{1}{2}$.

Formula (7) gives the value of $P(x+iy)$, the two terms being respectively the real and imaginary parts. It helps memorize (7) to note that the imaginary part of $P(x+iy)$ must contain the factor y, since its sign must depend on the sign of y.

4. Arrangement of the computation. When the coefficients are in decimal form, the rows of (1) are more conveniently displayed as columns. The values of a_j go in the first column, with a_0 at the top. Next proceed with the quadratic synthetic division, noting that the multiplications by p and q, and the additions are done on the desk calculator, so that the second and third rows of (1) do not have to be written. The values of b_j are put in a second column. (See (9)). The value of $P(x+iy)$ is read from the two final values of b_j, using (7).

To find $P'(x+iy)$, fill in a column showing the exponents of the successive terms. The coefficients of $P'(z)$ are found as the product of each a_j by the exponent. This multiplication can be done mentally, as when $a_j=30$ and the exponent is 5. Or, the machine will handle the multiplication, as when $a_j=2.179$ and the exponent is 7. In fact, on machines equipped with "automatic multiplication," it is as easy to enter (7)(2.179) into the machine as to enter 2.179.

So $P'(x+iy)$ is computed from $P'(z)$ by the process explained for computing $P(x+iy)$ from $P(z)$. The resulting coefficients are called b_j', and are entered in a column. Note that $P'(z)$ is of degree $n-1$.

By Newton's Formula,

$$\Delta z = -\frac{P(x + iy)}{P'(x + iy)} = -\frac{b_n + iyb_{n-1}}{b'_{n-1} + iyb'_{n-2}}. \tag{8}$$

Example 1. Given that

$$z^6 - 2z^5 + 2z^4 + 2z^3 + 7z^2 - 5z + 9 = 0$$

has a root near $0.42+0.87i$, carry out one Newton correction.

Solution. The computation in tabular form is displayed in (9). $x=0.42$, $y=0.87$, $p=0.84$, $q=-0.9333$, and $n=6$.

(9)

	a	b	b'
6	1	1.	6.
5	−2	−1.16	−4.96
4	+2	+0.0923	−1.7662
3	+2	+3.16016	+9.14556
2	+7	+9.568391	+23.330665 = b_4'
1	−5	+0.088071 = b_5	−3.736672 = b_5'
	+9	+0.106810 = b_6	

$$\Delta z = -\frac{0.106810 + 0.088071(0.87)i}{-3.736672 + 23.330665(0.87i)}$$

$$= -0.002714 + 0.005760i$$

$$z = \frac{+0.42 \quad +0.87i}{+0.417286 + 0.875760i = \text{corrected } z}.$$

5. A generalization. The suggested method of substitution applies in the more general case of the equation

$$P(z) + F(z) = 0, \tag{10}$$

where $P(z)$ is a real polynomial and $F(z)$ is any analytic function. In applying Newton's Method, the substitutions $P(x+iy)$ and $P'(x+iy)$ can be made as in Example 1, while some other method may be used to find $F(x+iy)$ and $F'(x+iy)$.

Example 2. Find a root of (10), where

$$P(x) = z^6 + 3z^5 + 8z^4 + 12z^3 + 20z^2 - 35z + 180,$$

$$F(z) = 10iz.$$

Solution. After a few trials and corrections, there seems to be a root near $1.2+1.2i$. One correction, starting with this approximation is shown, in (11). $x=1.2$, $y=1.2$, $p=2.4$, $q=-2.88$.

(11)

	a	b	b'
6	1	1	6.
5	3	5.4	29.4
4	8	18.08	85.28
3	12	39.84	156.00
2	20	63.5456	168.7936
1	−35	2.77024	−281.72768
	180	0.31296	

	Function	Derivative
$P(z)$	$0.313 + 3.324i$	$-281.7 + 202.6i$
$F(z)$	$-12.000 + 12.000i$	$10.0i$
$P(z) + F(z)$	$-11.687 + 15.324i$	$-281.7 + 212.6i$

$$\Delta z = -\frac{-11.687 + 15.324i}{-281.7 + 212.6i}$$

$$\Delta z = -0.0526 + 0.0147i$$

$$x + iy = \frac{+1.2 \quad + 1.2i}{+1.1474 + 1.2147i = \text{corrected } z}.$$

ON A PROPERTY OF HARMONIC FUNCTIONS

H. B. MANN, Ohio State University

It is well known that an harmonic function has derivatives of all orders. Hence if (x, y) is a point in the interior of a region where $f(x, y)$ is harmonic and if $D_x f = f_x$, $D_y f = f_y$ the form

$$(D_x h + D_y k)^n f(x, y) = \phi_n(h, k) \tag{1}$$

is well defined. Here h and k denote arbitrary real numbers.

THEOREM. *If $f(x, y)$ is an harmonic function in a neighborhood of the point (x, y) then $\phi_n(h, k)$ is either identically 0 or it is an indefinite form.*

Proof. We have, since $(D_{xx}+D_{yy})f(x, y)=0$,

$$(D_x h + D_y k)^2 f = D_x(D_x h' + D_y k')f, \tag{2}$$

where $h' = h^2 - k^2$, $k' = 2hk$. The pair h', k' can take all pairs of values since the hyperbolas $h^2 - k^2 = h'$, $2hk = k'$ intersect for any values h' and k'.

We now prove the theorem by induction. The form (2) is obviously indefinite or identically 0. If n is odd then (1) is clearly indefinite or identically 0. If $n = 2n'$ then by (2)

$$(D_x h + D_y k)^{2n'} f = D_x^{n'}(D_x h' + D_y k')^{n'} f,$$

where h' and k' can take all values. Since $D_x^{n'} f$ is also an harmonic function the theorem follows by induction.

The theorem improves on the well-known result that an harmonic function cannot take a minimum or maximum in the interior of its domain of definition. The proof moreover is shorter and more direct than the known proofs of this property of harmonic functions.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND JOHN R. MAYOR, AAAS and University of Maryland

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THE ROLE OF MATHEMATICS IN AN INTEGRATED 9TH GRADE SCIENCE-MATHEMATICS COURSE

ALEXANDER CALANDRA, Washington University

The contemporary interest in the reexamination of the teaching of mathematics at the elementary and high school levels has, for the most part, developed into an appraisal of the significance of modern mathematics for the curriculum at these levels. While the relevance of this appraisal has considerable merit, it has tended to obscure the heuristic value of the relationship of mathematics to the sciences, particularly physics.

The traditional curriculum does not do any better in this regard, for most of the mathematics needed in the conventional physics course is developed in the 9th grade while the physics course is usually taken in the 12th. This raises the question of the advisability of developing a substantial amount of elementary algebra which is taught at the 9th grade level, with the aid of a parallel systematic treatment of elementary physics.

As far as the author has been able to find out, this approach has not been previously attempted in any school, although a few related projects have been undertaken. The one system in operation after a number of years is the Baldwin School Project, Bryn Mawr, Pa., which combines the teaching of mathematics and science at the 7th grade level. In this project, however, the physics which is taught as part of the science is not a systematic treatment but consists of specially selected topics.

A more sophisticated attempt, at the 10th grade level, combining algebra and physics was introduced at the Putney School in Vermont a number of years ago, but the effort was not continued. In the summer of 1957, the author introduced an integrated course in physics and algebra at the 9th grade level at Bayless High School. The objectives of this course, as far as the mathematics were concerned, were as follows:

(1) To provide the student in elementary algebra with a systematic body of scientific material which could be explored with the mathematical skills learned in algebra.
(2) To decrease the time between the learning of mathematical skills and their application.
(3) To stimulate the re-thinking of the teaching of mathematics and science at the high school level.

Procedure. This integrated algebra-physics course at the 9th grade level was given five days a week for two hours a day for one school year. The students were given credit for one year of science and one year of algebra. The subject matter for the first semester was primarily geometric optics, a topic which the

writer had found to be very useful in introducing physics to students with weak backgrounds in mathematics. Although the mathematical content was not greatly different from that in the conventional algebra course, it was extensively structured to provide a pattern which would make sense both in mathematics and physics.

A major modification in the approach to algebra consisted of the immediate introduction of ratio and proportion and a delay in the treatment of signed numbers. The thinking behind this reorganization was that the topic of ratio and proportion is extensively used in geometric optics and is easy to teach, while the topic of signed numbers is a fairly sophisticated concept which can be advantageously delayed until the students have had considerably more facility in the handling of literal numbers.

Of the unconventional topics introduced, it was found necessary to include a small amount of very simple training in geometry and trigonometry. In geometry, this was accomplished by the use of a few special theorems that gave the student some insight into similar triangles. In trigonometry the only function used was the sine, which was thoroughly explored in terms of the unit circle concept. The emphasis on the sine came about as the result of the discussion of Snell's Law dealing with the Refraction of Light and the Diffraction Law in connection with a simple procedure for measuring the wave length of light. Although the subject matter of the first semester was primarily geometric optics, there was a substantial amount of other science introduced at such points where the mathematical treatment made them relevant. Thus, the Inverse Square Law was introduced in optics and was immediately applied to problems in gravitation, acoustics, radioactivity, electricity and magnetism. To the extent that this was done, the physics pattern was structured by the mathematics rather than the other way around. The mathematics of the second semester was developed around the concept of functions with a very elementary but integrated treatment of graphs, tables and equations. The linear equation $y=mx+b$ was explored intensively with a large variety of examples but with special emphasis on problems in kinematics. The second semester spent a considerable amount of time on mechanics including problems in rotation involving calculations with moments of inertia which gave extensive practice with quadratic equations. After mechanics, the course was concluded with electricity and magnetism. Electricity was an excellent medium for drill with simultaneous equations in connection with Kirchoff's Laws, and this was developed in considerable detail to advantage.

After a year's trial at Bayless High School, where the course was run by one instructor in both areas of science and mathematics, the course was expanded to other institutions in a variety of different circumstances. Sufficient experience is now available to give the following conclusions with considerable confidence.

(1) The course can be successfully given in the ninth grade by a teacher of average competence, with an average (but not overly heterogeneous) student group provided lesson plans are made available in considerable detail.

(2) The course can be given at the eighth grade level with an above-average group of students with a skilled and enthusiastic teacher. Students in these groups must also be studying algebra.

(3) It is not necessary that the same teacher be involved in both the mathematics and science. The science teacher can work cooperatively with the mathematics teacher, informing the latter of the background needed at different times. Although some of the administrators had misgivings about this practice, it has worked out exceedingly well.

(4) There is no deficiency in the resulting mathematical training of the students in the program. In one of the first trials of this program in the St. Louis Public School system, the median grade of one group of students (above average) in this program on the Lankton Algebra Test corresponded to the 94th national percentile.

(5) Although the course has been well received by students of average ability, it does particularly well with students of above-average ability.

Future Plans. The first class in this integrated course enrolled 22 students in 1957–58. In 1958–59 the venture was expanded to 200 students in 7 classes. The best estimates that can be made of enrollment in 1959–60 is about 5000 students in 250 classes. The actual figure will depend largely on the continued assistance of the United States Office of Education and possibly additional assistance from foundations.

The course is not an isolated science course, but part of a pattern of integrated science and mathematics throughout the secondary school system. It is part of a plan in which this course is followed by a sophomore chemistry course which emphasizes mathematical skills. This in turn is followed by a junior biology course which takes advantage of physics and chemistry previously learned and uses mathematical techniques in the field of statistics and probability in connection with the study of heredity and genetics. In the 4th year the science course planned is a second year of physics whose level of sophistication is comparable to the present typical college freshman physics course, and contains the kind of modern physics which is being presented in the Continental Classroom course in physics on television.

WISCONSIN PHYSICS FILM EVALUATION PROJECT

Progress Report

W. A. Wittich, The University of Wisconsin

The following is a brief summary of what appears in the January progress report.

"In September 1956 Dr. Harvey White began to teach the first one-year television course in high school physics ever to be offered as a continuing day-to-day experience in master teaching through television means. The original 30-minute telecasts were made over educational television station, WQED, in

Pittsburgh. One year later the University of Wisconsin undertook to evaluate the effectiveness of the Harvey White telefilm materials as they might be used in on-going, representative high school physics study situations in Wisconsin communities.

"The study undertook to discover (1) the effectiveness of the telefilms in typical classroom situations and (2) the effectiveness of telefilms in locally supervised correspondence study situations."

The January report is concerned only with the classroom phase of the study since the correspondence study phase has not been completed and is going on at the time of the writing of this summary.

The progress report states that:

"Television films can be described as a hybrid. They are made under rigid 30-minute television format conditions and therefore include many of the repetitions or unobtrusive errors or sometimes lengthy explanations which are characteristic of television *per se*. However, a limited amount of post judgment and improvement in quality is possible through later editing. This has been the case in the Harvey White physics films where the two media, sound motion picture film and television, have been brought into inter-play to create a new kind of educational material, the 16 mm. sound television film or T-film."

The report goes on to state that in September 1957, 83 schools began their participation in the T-film-using and control groups during the 90-day first semester of 1957–58 experimental period. All school groups, T-film-using as well as control, proceeded to study units on mechanics, properties of matter and heat during this 90-day semester.

The report describes how the development of the supplementary study materials, manuals for use by the instructors, tests for use in evaluation, and student supplementary study materials, were made through the cooperative efforts of Wisconsin State College Physics Department staff members. It describes the physics project as a high level, far-reaching, coordinated attempt by many of the educational teacher-training agencies in Wisconsin.

The report includes a series of questions and answers concerned with the key findings among traditional classroom groups which used the telefilms as experimental materials. Such questions as these were included:

Question: How did the use of the 30-minute film per day as measured by the Wisconsin final physics test influence the learning of students enrolled in high school physics classes?

Answer: The control and T-film using groups showed no significant difference in the amount of physics learned that was presented in both T-film and textbook.

The T-film-using group showed a significant level of achievement above that accomplished by control groups in the case of the physics being included in the Harvey White T-films alone.

Question: What influence did the use of a 30-minute Harvey White T-film have on the teacher's knowledge of physics?

Answer: The teachers who saw and used the film learned a significant amount of physics information above that learned by control teachers.

Question: How did the T-film using group compare with the control group in terms of retention of physics information?

Answer: The control group retained more Semester 1 physics information than the T-film-using group after a period of three months. Measurement was made on the basis of physics information found both in textbooks as well as in films.

It is believed by the committee that the Wisconsin Physics Film Project will have far-reaching implications and influences in the conduct of classroom teaching of physics. Information to be found in T-films only was tested for.

Interested persons may secure copies of the phase I report by addressing W. A. Wittich, 113 Education Building, University of Wisconsin, Madison. The committee which guides the Wisconsin project includes Professor Milton O. Pella, Department of Education, Professor Charles Wedemeyer, Director of Correspondence Study, Extension Division, and Professor W. A. Wittich, Department of Education.

THE UNIVERSITY OF ILLINOIS ARITHMETIC PROJECT

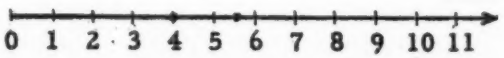
DAVID A. PAGE, University of Illinois

The University of Illinois Arithmetic Project was created in the fall of 1958. It is directed by Professor David A. Page. The project interprets the term "arithmetic" broadly and is actually concerned with many kinds of possible content in elementary mathematics classes. In November the Carnegie Corporation of New York announced a grant of $307,000 to the project for five years of work. Over the five year period the staff of the project plans to create an entirely new elementary mathematics course including textbooks, guides for teachers, and suitable teaching aids. Also they will be concerned with helping teachers learn to teach the new course.

An example of a classroom topic that has already been used in the experimental classes in Champaign-Urbana will give some idea of the kinds of new material which the project will be developing. The topic is "number line games and stand-still points." Quasi-elementary school language is used in order to substantiate the possibility of communicating with 4th and 5th grade students.

Number line games and stand-still points.

Here is a *number line:*



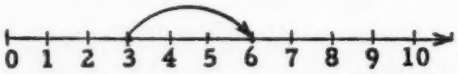
Think of it as going on to the right forever. On a number line every number (of the kind we work with) has its place. On the number line above

the places for 4 and $5\frac{1}{2}$ are shown by dots.

We will now learn about number line games. There are different number line games. In order to play one you must have a rule. Here is a rule for one number line game.

We will read this rule by saying, "Box goes to box times 2." This rule tells you how to make a *move* on the number line. First you must have a starting point. Suppose you start at 3. Put a 3 into each box of the rule. This gives you

The rule now tells you, "3 goes to 3 times 2." That is, it tells you that if you start at 3 and make one move, you go to 6. We can show this jump from 3 to 6 on a number line this way:



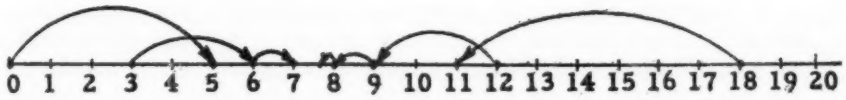
Where would you jump to if you started at 5 and made one move? Where would you jump to if you started at 2 and made 3 moves? Where would you have to start in order to get to 10 in 2 moves?

At this stage students can be given an enormous amount of computational practice appropriate to their grade level by varying the complexity of the rule and the "difficulty" of the starting point. Instead of indicating the wide variety of topics which can be taken up next, a specific one, of stand-still points, will be considered.

Take another number line game with this rule

(*)

We read this rule, "Box goes to box divided by 3, plus 5." If we choose several starting points and show the jump that this rule tells us to make for each of the starting points on a number line, it looks like this:



(Remember that these students have no algebraic technique for solving such a problem.) Here, in some abbreviation, are various conjectures and observations that students made in their search for the stand-still point:

1) In one part of the number line all of the jumps are to the right and in another part all the jumps are to the left. If I could find the place where the jumps stop going to the right and start going to the left I would have the stand-still point.

2) When you start at some points on the line you make long jumps. When you start at other points on the line you make short jumps. If you are at the stand-still point you don't jump at all. So if I keep heading toward shorter and shorter jumps I will be headed toward the stand-still point.

3) Look at the picture. One jump takes me from 3 to 6. Another jump takes me from 12 to 9. Since these jumps have the same length but are in opposite directions, it seems that the stand-still point should be halfway between the two starting points. It should be $7\frac{1}{2}$.

4) It seems to me that wherever I start on the number line my jumps carry me toward smaller jumps and toward the jumps in the opposite direction. So no matter where I start jumping on the number line, I always get closer to the standstill point. Therefore, if I pick any old point as a starting point and keep making one jump after another, I am bound to get to the stand-still point. (In practice, if the stand-still point is an "easy" one, students applying this "just keep jumping" method make a few successive jumps and then guess the stand-still point correctly. Cases have come up, however, where students recognized that although they kept getting closer and closer to the stand-still point, they would never be able to reach it.)

5) Look at the rule(*) above. At the far right you add 5. In order to stand still, you must lose 5. The only place you can lose 5 is when you divide by 3. So you are looking for a number such that when you divide it by 3, the result is down 5 from what you start with. Try something, say 15. Fifteen divided by 3 is 5 and so you have gone from 15 down to 5 or down 10. Too much. Try 9. Nine divided by 3 is 3 and you have gone from 9 to 3 or down 6. Still too much but getting closer. Try 6. Too little. Keep trying. Try $7\frac{1}{2}$. Seven and one-half divided by 3 is $2\frac{1}{2}$ and going from $7\frac{1}{2}$ to $2\frac{1}{2}$ is going down by 5. I must have the stand-still point.

6) My way is like the one before but faster. Whenever your rule is to divide by 3 and add something, take that something and divide it by 2. Take what you get and add it to the something and you have the stand-still point.

7) Another student made a two-dimensional graph with one axis an axis of starting points and the other axis the axis of landing points (he did not use this language!). Thus one point on his graph corresponded to a complete jump for the game. From this graph he was able to read off rather easily the location of stand-still points.

The reader can correctly infer from the responses indicated above that these students were of high ability. The project is far from ashamed when it discovers material which challenges and interests bright students. However, in the long run, the project is concerned at least with students of average ability and above and perhaps with students of below average ability. Able students were needed in order to find so many approaches to the job of finding stand-still points which can in turn be presented in a more gradual way to less able students.

AAAS TWO-YEAR STUDY ON THE USE OF SCIENCE COUNSELORS

J. A. Brown, University of Delaware

The AAAS Study on the Use of Science Counselors, developed as a result of the awareness of the present and anticipated future shortage of well-qualified science and mathematics teachers, is based on the recognition of the valuable service that can be provided to teachers by competent supervisors or counselors. Four centers for study were designated to represent the west, the south, the midwest, and the east. These centers were the Universities of Nebraska, Oregon, Texas, and Pennsylvania State University. In each university the study had the full support of the science, mathematics, and education departments. The universities were responsible for establishing satisfactory relationships with the cooperating schools and the state departments of education. Two experienced teachers, each capable of functioning as a counselor in science and mathematics teaching, were employed by the universities with funds made available to AAAS by a grant from the Carnegie Corporation of New York.

The direct-contact work of the counselors was in five areas: subject-matter counseling, materials and methods counseling, curriculum counseling, career-guidance counseling, and teacher professionalization.

During the two years of the study, the counselors visited some 150 science and mathematics teachers in secondary schools in these regions. These schools varied in size and were dispersed geographically in the states. Counselors visited schools, sometimes as a team and sometimes as individuals. They sought to bring new ideas, current thinking, and recent developments in science and mathematics teaching directly to the teachers. Group sessions for science and mathematics teachers were held for teachers from time to time, both on the university campuses and in cooperating schools.

The evaluation of the study under the supervision of John W. Gustad, Professor of Psychology of the University of Maryland, was first reported to the annual AAAS meeting in December. The evaluation involved questionnaires

and interviews. A total of 700 questionnaires were sent to participating teachers, with a 91 per cent return. Data were obtained from counselors, and interviews were conducted with a total of 60 individuals, including teachers, administrators, university staff members, and representatives of state departments of education. Gustad reported that virtually all expressed appreciation for the program and a desire to see it continued; and, in addition, all expressed a desire for substantially more contacts with counselors under a new program. Teachers as a group are moderately well satisfied with the support they receive from administrators and also with administrators' understanding of the problems of science teaching. Some indicated that the Study activity assisted on this point. There was general agreement among all groups that the study had stimulated the interest of the teacher in his own field, had provided him with new ideas about teaching methods, created new interest in science on the part of the library staff, assisted the guidance counselor, and stimulated greater student interest in science. Questionnaire and interview responses emphasized the desire of the teachers for more training in science subject matter and the opinion that participation in professional societies in science was more valuable to them than participation in professional education organizations.

A final report of the Study may be obtained upon request addressed to the Director of Education, AAAS, 1515 Massachusetts Ave. N.W., Washington 5, D. C.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1366. *Proposed by V. E. Hoggatt, Jr., San Jose State College*

Show that if a, b, c form a triangle, then $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ form a triangle.

E 1367. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Prove that the numerator of the sum $1-1/2+1/3- \cdots$ to $[2p/3]$ terms is divisible by p when p is a prime greater than 3.

E 1368. *Proposed by M. S. Klamkin, AVCO Research and Development*

Show that if all roots of $ax^4-bx^3+cx^2-x+1=0$ are positive, then $c \geqq 80a+b$.

E 1369. *Proposed by R. E. Shafer, University of California*

If $y=(1+x^{1/2})^{2n+2}$, show that $y^{(n)}=4(n+1)(n+1)!$ at $x=1$.

E 1370. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

(a) Evaluate $\int_0^\infty (1-e^{-x})^n x^{-n} dx$ for integral $n>1$.

(b) Evaluate $\int_0^\infty (1-e^{-x})^2 x^{-2} \cos^2 kx\, dx$ for arbitrary $k>0$.

SOLUTIONS

A Theorem of Darboux Type

E 1336 [1958, 708]. *Proposed by Albert Wilansky, Lehigh University*

Let $f(0)>0$, $f(1)<0$. Prove that $f(x)=0$ for some x under the assumption that there exists a continuous function g such that $f+g$ is nondecreasing.

Solution by the Proposer. Let E be the set of x such that $f(x)\geqq 0$. Let $s=$ l.u.b. E. Let $h=f+g$. Then, since h is nondecreasing, for any $x\in E$, $h(s)\geqq h(x)\geqq g(x)$. Hence, since g is continuous, $h(s)\geqq g(s)$ and so $f(s)\geqq 0$.

Next, $g(1)>h(1)\geqq h(s)\geqq g(s)$. Since g is continuous, there exists $t\geqq s$ such that $g(t)=h(s)$. Then $h(t)\geqq h(s)=g(t)$, so that $f(t)\geqq 0$. By definition of s, $t=s$ so that $g(s)=h(s)$ and $f(s)=0$.

Also solved by H. F. Bechtell, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), C. H. Cunkle, S. J. Einhorn, Michael Goldberg, L. C. Graue, R. E. Greenwood, L. E. Hanson, Lawrence House, Joe Lipman, D. C. B. Marsh, James Misho, L. E. Pursell, Hans Schneider, Robert Spira, G. B. Thomas, Jr., and Dale Woods. Late solutions by J. W. Baldwin, D. A. Freedman, A. R. Hyde, and Jack Silver.

Generalization of a Russian Olympiad Problem

E 1337 [1958, 708]. *Proposed by the Student-Faculty Colloquium, Carleton College*

What is the largest value of n, in terms of m, for which the following sentence is true? If from among the first m natural numbers any n are selected, among the remaining $m-n$ at least one will be a divisor of another.

Solution by D. D. Strebe, University of South Carolina. Let $n=[m/2]-1$. The remaining $m-n$ numbers contain the integer 1 or a pair of the form $(k, 2k)$, satisfying the divisibility requirement. To show that this is the largest value of n, select a subset consisting of the first $n+1$ natural numbers. In the remaining set of $m-(n+1)$ natural numbers, each number is greater than one half of any other number in the set; thus there are no divisors.

Also solved by R. G. Albert, Merrill Barnebey, Lawrence Botsford, R. F. Brown and Joel Levy (jointly), S. J. Einhorn, E. L. Ellis, E. A. Fay and R. S. Gardner and T. L. Reynolds (jointly), R. E. Greenwood, J. H. Hodges, Joe Lipman, D. C. B. Marsh, J. B. Muskat, J. L. Selfridge, Charles Wexler, and the proposers. Late solutions by J. W. Baldwin, D. A. Breault, Helen M. Marston, and W. A. Veech.

Editorial Note. See problems 3739 [1937, 120] and 3820 [1939, 240].

A Property of Two Projectively-related Point Rows

E 1338 [1958, 708]. *Proposed by V. F. Ivanoff, San Carlos, California*

Let P and P' be a pair of corresponding points of two projectively-related point rows. If the rows are inverted about P and P', respectively, show that the resulting point rows are similar.

Solution by L. D. Goldstone, New York State Public Works Laboratory, Albany, N. Y. Let Q, R, S be any three points on the point row containing P, and let Q', R', S' be the projectively-related points on the point row containing P'. Let P_i, Q_i, R_i, S_i denote the inverses of P, Q, R, S, and let P_i', Q_i', R_i', S_i' denote the inverses of P', Q', R', S'. Since cross-ratio is preserved under inversion we have

$$(P_iQ_iR_iS_i) = (PQRS) = (P'Q'R'S') = (P_i'Q_i'R_i'S_i').$$

But, since P and P' were the centers of inversion, P_i and P_i' are both ideal points. It now follows that $Q_iS_i/Q_iR_i = Q_i'S_i'/Q_i'R_i'$, and the theorem is established.

Also solved by Joe Lipman, D. C. B. Marsh, and the proposer.

"Un Calcul Simple"

E 1339 [1958, 708]. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

Let $f(t) = t - t^3/6 + (t^4/24) \sin(1/t)$ for $t > 0$. Prove that for $x > 0$, $z > 0$, $x + z < 1$, the relation $f(x+z) \leqq f(x) + f(z)$ holds.

Solution by D. C. B. Marsh, Colorado School of Mines. Define

$$g(t) = f(t)/t = 1 - t^2/6 + (t^3/24) \sin(1/t)$$

and note that for $0 < t < 1$

$$g'(t) = -(t/24)[8 - 3t \sin(1/t) + \cos(1/t)] < 0.$$

Therefore $g(t)$ is decreasing in this interval, and for $x > 0$, $z > 0$, $x + z < 1$, we have $g(x+z) < g(x)$ and $g(x+z) < g(z)$, which imply that

$$xg(x+z) + zg(x+z) < xg(x) + zg(z)$$

or $f(x+z) < f(x) + f(z)$.

Also solved by A. F. Kaupe, Jr., and the proposer. Late solution by C. H. Cunkle.

The proposer pointed out that this problem is equivalent to "un calcul simple" omitted from M. J. Haantjes' paper (see p. 91) *Sur la géométrie infinitésimal des espaces métriques* (Colloque de Géométrie Différentielle, Louvain, 1951).

A Type of Periodicity for Fibonacci Numbers

E 1340 [1958, 708]. *Proposed by Peter Beisswanger, Tübingen, Germany*

Let b_n be the number given by the last $k+2$ digits of a_n, where a_n is the nth

term of the Fibonacci sequence $a_0=0$, $a_1=1$, $a_{n+1}=a_{n-1}+a_n$. Show that for each $k\geqq 1$, the sequence $\{b_n\}$ is periodic with $150\cdot 10^k$ as a period.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* The b_n are congruent to the a_n (mod 10^{k+2}). Since the a_n have periods of $3\cdot 2^{m-1}$ (mod 2^m) and $4\cdot 5^m$ (mod 5^m), the b_n have period of l.c.m. $(3\cdot 2^{k+1}, 4\cdot 5^{k+2})=150\cdot 10^k$.

II. *Solution by Peter Greiner, University of British Columbia.* By mathematical induction one may establish the following two lemmas:

LEMMA 1. $(1+\sqrt{5})^n=2^{n-1}(a+b\sqrt{5})$, where a and b are integers with $a+b$ even.

LEMMA 2. $(1+\sqrt{5})^{150\cdot 10^k}=2^{150\cdot 10^k}[(10^{k+2}e+1)+10^{k+2}d\sqrt{5}]$, where d and e are integers and $k\geqq 1$.

One may now easily show that

$$a_n=[(1+\sqrt{5})^n-(1-\sqrt{5})^n]/2^n\sqrt{5}=b,$$

$$a_{n+150\cdot 10^k}=10^{k+2}(ad+bc)+b,$$

whence

$$a_{n+150\cdot 10^k}-a_n=10^{k+2}(ad+bc).$$

Also solved by Vern Hoggatt, Joe Lipman, Walter Penney, and the proposer.

A solution of the problem may be found in Vern Hoggatt, *A type of periodicity for Fibonacci numbers*, Mathematics Magazine, Jan.–Feb. 1955, pp. 139–142. The problem arose in a research by Hoggatt concerning the existence or nonexistence of three Fibonacci numbers which are squares of the sides of a right triangle. This latter problem is still unsolved.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4845. *Proposed by Fulton Koehler, University of Minnesota*

Let α be a real square matrix $[a_{ij}]$ and let $D(\alpha)=\sum_{i\geqq j}a_{ij}$. Find the maximum of $D(\alpha)$ for the group of orthogonal matrices of given order n; and show that, as $n\to\infty$, the maximum is asymptotic to $(n/\pi)\log n$.

4846. *Proposed by Olga Taussky, California Institute of Technology*

Show that a matrix which is similar to a real diagonal matrix is the product of two hermitian matrices one of which is positive definite. (The converse is known.)

4847. *Proposed by P. T. Bateman, University of Illinois*

Suppose f is a nonconstant polynomial function over a field K. Show that the additive group of K is generated by the values taken on by f provided either (a) K is of characteristic zero or (b) K has prime characteristic p and f has degree less than p. Also show by an example that some restriction such as (a) or (b) is needed.

4848. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Massachusetts*

Without performing any integration determine the ratio

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} \quad : \quad \int_0^1 \frac{dt}{\sqrt{1+t^4}}.$$

4849. *Proposed by L. A. Rubel, Institute for Advanced Study*

Suppose that $\phi(t)$ is a bounded measurable real function defined for $t>0$, and let $f(x)=(\phi * \phi)(x)=\int_0^x \phi(t)\phi(x-t)dt$. Must $f(x)$ be nonnegative for all sufficiently small positive x?

4850. *Proposed by Ward Cheney and Allen Goldstein, Convair-Astronautics, San Diego, California*

Let $\mathfrak{A}$ and $\mathfrak{B}$ denote two closed convex sets in Hilbert space. For a point x, let Ax and Bx denote respectively the points of $\mathfrak{A}$ and $\mathfrak{B}$ closest to x. Prove that A satisfies the Lipschitz condition $\|Ax-Ay\| \leqq \|x-y\|$. Furthermore, if x is a fixed point of AB, then $\|x-Bx\| = \text{dist}\ (\mathfrak{A}, \mathfrak{B})$.

SOLUTIONS

An Invalid Conjecture

4673 [1956, 125]. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Let $a_1<a_2<\cdots$; $b_1<b_2<\cdots$ be two sequences of integers. Prove that from the sequence a_i+b_j one can always select an infinite subsequence such that no one divides another.

Note by the proposer. The proposition is not true as a counterexample shows. Trostrum, in *Mathematica*, June 1958, constructed two sequences a_i, b_i, $1 \leqq i < \infty$ such that amongst the integers a_i+b_j there do not exist infinitely many, no one of which divides another.

Continuous Function with Rational Values Almost Everywhere

4799 [1958, 530]. *Proposed by K. L. Chung, Syracuse University*

Find a function continuous in $(0, 1]$ without any interval of constancy which takes on rational values almost everywhere.

Solution by Robert Spira, Berkeley, California. Define $f_{10}(x)$ as $1/2-|x-1/2|$. At the midpoint of each slanted portion of the graph insert an interval of constancy of length 3/8 and having the same midpoint. Reconnect the ends of the slanted portion to the ends of the interval of constancy and delete the old slanted portion. The result is the graph of $f_{11}(x)$. Now replace the center part of each horizontal portion with an inverted V, of base 1/16 and height 1/32. This is $f_{12}(x)$. There are now four disjoint horizontal sections; at the center of each put an inverted V with base 1/64 and height 1/128; the total of the four bases is 1/16. This corresponds to $f_{13}(x)$. Continue on in this manner; in general, the graph of $f_{1n}(x)$ will have 2^n horizontal sections, and the total sum of the bases of the inverted V's which have been added since the first step will be $(1/8+1/16 + \cdots +1/2^{n+1})$.

The $f_{1n}(x)$ converge uniformly to a continuous function $f_1(x)$ which takes on the value 1/4 on a nondense set of measure 1/2.

For $f_{20}(x)$ take $f_1(x)$. On each slanted portion, insert at the midpoint an interval of constancy with length 3/4 of the difference of the abscissas of the endpoints of the slanted portion. Connect the endpoints of each interval of constancy to its respective endpoints of the slanted portion. This is the graph of $f_{21}(x)$. Altogether, the intervals of constancy of $f_{21}(x)$ add up to 3/8.

Each interval of constancy of $f_{21}(x)$ has length $(3/2)\cdot(1/2^m)$. In any such interval, in a manner analogous to the previous construction, a nondense set may be formed as the limit of $f_{2n}(x)$; where the total measure of the added inverted V's is $(1/2)\cdot(1/2^m)$. Thus $f_2(x)=\lim f_{2n}(x)$ has rational values on a set of measure 3/4.

Going on in this fashion, there is determined a sequence of continuous functions $f_1(x), f_2(x), \cdots$. These functions converge uniformly to some $f(x)$, which is continuous and takes on rational values on a set of measure 1, and has no interval of constancy.

Also solved by Jan Lipinski and by S. P. Lloyd.

Representation as Sums of Squares

4801 [1958, 530]. *Proposed by S. W. Golomb, Pasadena, California*

Let $R(n)$ denote the minimum number of terms required in representing n as a sum of squares. If n is restricted to perfect numbers, show that

$$R(n) = \begin{cases} 2 \text{ for odd } n, & \text{if any;} \\ 3 \text{ for } n = 6; & \\ 4 \text{ for even } n, & n \neq 6. \end{cases}$$

Solution by P. T. Bateman, University of Illinois. First, suppose n is an odd perfect number. Then (Euler) $n=p^rm^2$, where p is prime, m is not a multiple of p, and $p\equiv r\equiv 1 \pmod 4$. Thus n is not a square but there exist integers a, b (Euler) such that $p=a^2+b^2$. So $R(n)=2$. In fact

$$n = \{p^{(r-1)/2}ma\}^2 + \{p^{(r-1)/2}mb\}^2.$$

Clearly $R(6)=3$.

Finally, suppose n is an even perfect number other than 6. Then (Euler) $n=2^{p-1}(2^p-1)$, where p is an odd prime. Thus $n=4^{(p-1)/2}(2^{p-3}\cdot 8-1)$, which is of the form $4^a(8b+7)$, and hence (Fermat) n is not expressible as a sum of three squares. Finally (Lagrange) $R(n)=4$.

For the theorems referred to, one may consult Landau, *Elementary Number Theory*, New York, 1958; theorems 32, 165, 169, 186, and ex. 4, p. 235.

Also solved by Anders Bager, J. L. Brenner, L. Carlitz, Emil Grosswald, J. H. Hodges, M. I. Knopp, J. B. Muskat, Benjamin Sapolsky, Robert Spira, M. Sugunamma, and the proposer.

Lower Bound for the Maximum Element in a Matrix

4802 [1958, 530]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let $A=(a_{ij})$ be a real symmetric matrix of order n ($\geqq 3$) and of rank $\leqq 3$. If A is positive, semi-definite and if $a_{ii}=1$ ($1\leqq i\leqq n$), then

$$\operatorname*{Max}_{i\neq j} a_{ij} \geqq \tfrac{1}{2}\operatorname{cosec}^2 \omega_n - 1,$$

where $\omega_n=n\pi/6(n-2)$.

Solution by the proposer. The positive semi-definite matrix A can be expressed as $A=B^2$, where B is a real symmetric matrix of order n. Let b_i denote the ith row vector of B. Then the inner product $(b_i, b_j)=a_{ij}$ ($i, j=1, 2, \cdots, n$). Since $a_{ii}=1$, we have $\|b_i\|=1$. Since A is of rank $\leqq 3$, the maximum number of linearly independent vectors among $b_1, b_2, \cdots, b_n$ is $\leqq 3$. Hence the points $b_1, b_2, \cdots, b_n$ lie on the unit sphere S^2 of a 3-dimensional linear subspace of the Euclidean n-space. Now, by a theorem of L. Fejes Tóth and H. Hadwiger, among any n ($\geqq 3$) points on the unit sphere S^2, there is at least one pair of distinct points with distance $\leqq(4-\operatorname{cosec}^2\omega_n)^{1/2}$. [See L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und in Raum*, Springer, 1953, p. 115; also W. Habicht and B. L. van der Waerden, *Lagerung von Punkten auf der Kugel*, Math. Annalen, 123 (1951), pp. 223–234.] It follows that

$$\operatorname*{Min}_{i\neq j} \|b_i - b_j\|^2 \leqq 4 - \operatorname{cosec}^2 \omega_n.$$

As $\|b_i\|^2=1$, this means $\operatorname{Max}_{i\neq j} a_{ij}=\operatorname{Max}_{i\neq j}(b_i, b_j)\geqq \frac{1}{2}\operatorname{cosec}^2\omega_n-1$.

A direct proof without using the theorem of Fejes Tóth and Hadwiger is greatly desired. It is also hoped that a generalization by considering arbitrary upper bound r (instead of 3) for the rank of A will be discussed.

Series with Negative Coefficients

4803 [1958, 530]. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Let $\sum a_n x^n$ be analytic at 0 and suppose $a_n > 0$, $a_{n+1}a_{n-1} > a_n^2$. Prove that the expansion of $1/\sum a_n x^n$ has all negative coefficients (except for the constant term).

Solution by Leonard Carlitz, Duke University. This result is due to Kaluza, *Mathematische Zeitschrift*, vol. 28, 1928, pp. 161–170, in particular, Theorem 3. The simple proof follows. We may assume $a_0 = 1$. Put

$$1 \Big/ \sum_{n=0}^{\infty} a_n x^n = 1 - \sum_{n=1}^{\infty} c_n x^n,$$

so that

$$0 = a_n - \sum_{r=1}^{n} c_r a_{n-r} \qquad (n \geqq 1), \tag{1}$$

$$c_{n+1} = a_{n+1} - \sum_{r=1}^{n} c_r a_{n-r+1}. \tag{2}$$

If we multiply (1) by $-a_{n+1}$ and (2) by a_n, and then add, we get

$$a_n c_{n+1} = \sum_{r=1}^{n} c_r (a_{n+1}a_{n-r} - a_n a_{n-r+1}).$$

Now the hypothesis $a_{n+1}a_{n-1} > a_n^2$ implies $a_{n+1}a_{n-r} > a_n a_{n-r+1}$ $(1 \leqq r \leqq n)$. Hence, by induction every $c_n > 0$ and the theorem follows.

Other references are: Szegö, *Math. Z.*, vol. 25, p. 177; Hardy, *Divergent Series*, Th. 22, pp. 68–69.

Also solved by A. C. Aitken, P. T. Bateman, O. Buchta, W. S. Lawton, D. C. Russell, W. F. Trench, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

College Geometry. By L. H. Miller. Appleton-Century-Crofts, New York, 1957. x+201 pp. $4.50.

This is not "just another college geometry." The author has incorporated much of the spirit and some of the content of projective geometry. It is an

elementary text presupposing high school plane geometry, elementary trigonometry, and familiarity with the concept of a limit.

Chapter 1 (27 pp.) is concerned with the nature of proof and a review of elementary theorems from plane geometry. In Chapter 2 (19 pp.) directed line segments and angles are used to introduce the ideal point on a line (the point which divides a line segment in the ratio -1), cyclic quadrilaterals, and the Theorems of Menelaus and Ceva.

Chapter 3 (11 pp.) is an introduction to constructions with ruler and compass. Chapter 4 (11 pp.) is based on six theorems (including the Circle of Apollonius) involving geometric loci. Chapter 5 (18 pp.) is based on transformations —line reflections (orthogonal), point reflection, translation, rotation, homothetic transformation (treated in terms of a change of scale with a coefficient of expansion). Similar and homothetic figures are included along with homothetic constructions.

Chapter 6 (22 pp.) involves inverse points and curves. The construction of a circle tangent to three given circles, orthogonal circles, the invariance of angles under inversion, and Ptolemy's Theorem are considered.

Chapter 7 (25 pp.) includes double-ratio (cross-ratio), central projection (perspectivity between planes), a trigonometric proof of the invariance of double ratios under a central projection; harmonic range (set of 4 points), construction of fourth harmonic, pencil of lines (defined as any set of concurrent lines), harmonic pencil, complete quadrilateral (tacitly assumed to be a plane figure), complete quadrangle, principle of duality (planar), pole and polar with respect to a circle (in terms of inverse points and perpendicularity), Desargues' Theorem, and straight-edge constructions. This reviewer considers the inclusion of concepts from projective geometry very desirable, the minor deviations from standard terminology (as indicated by some of the above parenthetical remarks) unfortunate but not serious.

Chapter 8 (23 pp.) includes radical axis, power of a point with respect to a circle, coaxial circles, the nine-point circle, Euler line, inscribed and escribed circles, Feuerbach's Theorem, Pascal's Theorem, isotomic and isogonal lines, and Miquel point.

Chapter 9 (8 pp.) is based on Steiner's Theorem.

Chapter 10 (15 pp.) involves the construction of special quadrilaterals.

Chapter 11 (13 pp.) is concerned with concurrent lines; Chapter 12 (5 pp.) with impossible ruler and compasses constructions.

Very few typographical errors were noted. The emphasis upon relations among theorems and concepts is commendable. The restriction to plane geometry is conventional. The modification of the traditional content of "college geometry" seems to this reviewer to be a much-needed move in the proper direction. Whether the change is sufficiently extensive will depend upon the tastes of the instructor. This reviewer considers the text to be a good start.

BRUCE E. MESERVE
Montclair State College

Introduction to Multivariate Statistical Analysis. By T. W. Anderson. Wiley, New York, 1958. 374 pp. $12.50.

This is an excellent introductory text. To read it one needs to know the calculus, including the calculus of functions of several variables, elementary matrix theory, including characteristic vectors and roots, and the contents of a course in probability and mathematical statistics having the calculus as prerequisite and going through elementary regression theory, limiting distributions, and an introduction to statistical inference including such notions as maximum likelihood estimates, power, invariance, Bayes and minimax estimates and decision theory. It is written for statisticians.

The topics covered are well indicated by the chapter headings. After the introductory chapter come chapters on the multivariate normal distributions; estimation of mean vector and covariance matrix; sample correlation coefficients; generalized T^2; classification; the distribution of the sample covariance matrix and generalized variance; the general linear hypothesis; independence of sets of variates, hypothesis of equality of mean vectors and covariance matrices; principal components, canonical correlations; characteristic roots and special topics.

There are many worked-out examples in the text and at the ends of the chapters appear a large number of carefully chosen exercises. A full bibliography is given.

The book emphasizes the distributional aspects of multivariate analysis. This is still the most developed part of the subject, and its value in applied work has been considerably increased by the use of electronic computers.

For those with an adequate background the book is well written and provides an excellent summarization of much of the work in this area.

The reviewer wishes that the index were more complete, and that the author had written a brief overview of statistical inference in the same spirit as that in which he provided an appendix on matrix theory. But these are minor quibbles in commenting on an excellent book.

WILLIAM G. MADOW
Stanford University and Stanford Research Institute

Introduction to the Theory of Sets. By Joseph Breuer (translated by Howard F. Fehr). Prentice-Hall, Englewood Cliffs, N. J., 1958. viii+108 pp. $4.25.

Helped on by Sputnik, more nonmathematicians (including many secondary school teachers) are interested in taking an additional mathematics course. The real difficulty is to provide a worthwhile course, something that does not require calculus, something that is "new" and interesting. Harold Franklin Fehr's translation of Joseph Breuer's *Introduction to the Theory of Sets* provides an excellent text for the introduction to real and "new" mathematical ideas to the nonspecialist. Set theory, by adding to the concept of finite number the con-

cepts of transfinite numbers, should add to the appreciation of elementary mathematics.

A brief and accurate description of the aim of the book and of the manner in which the topics are developed is given by the translator in the Preface: "There is now a need for a treatment of set theory in English, from a less-than-abstract axiomatic approach, sufficiently elementary to serve as an introduction to the subject for college and high school instructors, college students, and interested laymen. This book meets that need. A naive approach, which depends upon observation of the concrete world for its development and meaning, is a natural way to introduce the subject, and this procedure is used in the following exposition. Little by little, certain properties and principles are developed, which in turn are used to prove further theorems concerning sets as collections of abstract entities. Thus one is led from concrete finite sets, to cardinal numbers, to infinite cardinals, and thence to ordinals via the use of ordinal-types. Abstract set theory based on an axiomatic system is not treated here. . . . The axiom of choice and its relation to the theorem of well-ordering have had tremendous effect on the whole development of set theory, but these are matters of concern to the mathematician rather than to the neophyte."

The chapters are: Finite Sets, Infinite Sets (cardinal numbers), Ordered Sets, Point Sets, Conclusion (paradoxes and a discussion of formalism and intuitionalism), and an Appendix (including a brief historical outline and a bibliography). All material is introduced first with examples, and most proofs that are given are directly based on an example or are illustrated in detail by one. In addition, at the end of each short section (*i.e.* every five or six pages) there is a short set of exercises asking about further examples or about details of previous explanations. Complete answers for all the exercises are given in the back of the book. Facts with proofs beyond the limited scope of the book (as the Well-Ordering Theorem or the nonexistence of sets M and N with M equivalent to no subset of N, and conversely) are stated, explained and used.

The inclusion of the historical material, and the discussion of different viewpoints in mathematical thought, together with the statement of several unsolved problems and the mention of areas where much is still unknown, are ways in which this book seeks to broaden the reader's outlook on mathematics.

There is some danger that the reader will feel that he is being "talked down to" in the early pages of the book and then in the closing section feel discouraged by the difficulty of understanding some comments or that he will miss the significance of them. This is particularly true in the section on Point Sets, offered as an application of set theory. This section is not as detailed in its presentation as the other material and contains many ideas that are stated and not pursued nor closely linked as applications. One hopes that the user of this book realizes the introductory and incomplete nature of much of the material and is spurred on to further study of axiomatic set theory.

LIDA K. BARRETT
University of Utah

Introductory Probability and Statistical Inference for Secondary Schools. (An Experimental Course, Preliminary Edition.) Prepared for the Commission on Mathematics, College Entrance Examination Board, 1957. x+182 pp. $1.00.

It is a pleasure to read this anonymous little book and to speculate on how a first course in statistics in college could maintain the interest which must surely be developed by such a high school course. The authors clearly expose the principal concepts in the modern approach to probability and statistical inference using a minimum of mathematical apparatus and using examples and exercises which are almost uniformly interesting in themselves and which should indicate to the student the potential practical usefulness of statistical inference.

After acclimatizing the reader with 42 pages on frequency distributions, histograms, ogives, means, standard derivations and Chebyshev's theorem, 28 pages are given over to the introduction of concepts in probability including sample spaces, events, independence, conditional probability. The next 20 pages go over these concepts again with definitions and theorems expressed more formally. For a 42 page introduction to statistical inference, the binomial distribution is introduced and used in testing hypothesis in acceptance sampling with emphasis on operating characteristic curves. The three appendices are described briefly below.

In the selection of topics to introduce the student to statistical inference, the present book is in many ways similar to S. S. Wilk's *Elementary Statistical Analysis* (Princeton University Press, 1948, 1951). All the topics in the present text are included in Wilks' and each major topic occupies about the same amount of space. Wilks' text is written for a college level course, is longer, and assumes more previous study of mathematics. The present text spends more space on fundamental concepts, using examples and exercises which develop the meaning of these concepts without getting into complex mathematical manipulations.

For the most part the authors have been careful to say what they mean. In most places where an advanced reader can detect ambiguities, the correct idea will be conveyed to the reader at whom the book is aimed. More nearly complete statements might well convey no meaning at all. The most serious error the reviewer found was in attributing the use of cream to Fisher's tea-taster. Of course, she used milk in her tea.

The mathematical prerequisites to an understanding of the text include knowledge of roots of quadratic equations, some manipulative ability with inequalities, and some understanding of the relation between single valued functions and their graphs. Two other prerequisites to the main body of the text are covered in appendices: elementary set theory and permutations and "selections." A third appendix treats mathematical induction. The appendices are more complete than necessary for the text proper and one might suspect that the authors have some convictions as to what every high school student should know about mathematics.

College students of statistical inference will be pleased to find a book in

which fundamental concepts are made clear. The present reviewer is incompetent to guess the number of high schools which will in the near future add a course in probability and statistical inferences to their offerings. Those which do, and we can hope that the number is large, will find the present text excellent.

KENNETH J. ARNOLD
Michigan State University

Initiation à la Logique. (Collection de Logique Mathématique, Série A, XIII.) By Le R. P. Dubarle. Paris, Gauthier-Villars, 1957. 90 pp. 1400 fr. (about $3.50).

The author calls this an "inventory and classification" of the principal subjects of mathematical logic for those seeking only an introduction and a minimal acquaintance with the field. The book is divided into three major sections. The first attempts to explain the spirit of symbolization that led from classical logic to contemporary symbolic logic and the relation of this change to the formalization of mathematics. The author's philosophical orientations reveal themselves in these pages. He views the whole development as due to a nominalistic interpretation of mathematical entities. The second section provides the inventory of the concepts of mathematical logic, which have become so widely used in courses in mathematical foundations and modern algebra. The outline ends with a reference to multivalued logic, intuitional logic, modal logic, natural deduction, and combinatory logic. The final section discusses the limits of the implications of logic and mathematics and refers to Gödel's and Church's theorems on incompleteness and consistency.

It is difficult to evaluate this book. Although it achieves its aim of being an inventory quite adequately, it has all the defects of an inventory.

One point of Dubarle's is worthy of being underlined. Formalism is valuable, but its function is to represent results already achieved. Its use in "discovering" proofs is quite limited.

Typographical errors: Page 43, the definition of the existential quantifier has "$E(x)A(x)$" and should read "$(Ex)A(x)$."

The axiom 2.22 has "$\sim(x)A(x)$" and should read "$\vdash(x)A(x)$."

Page 84, the reference to Tarski should read "Einführung" and not "Fin Führung."

L. O. KATTSOFF
Harpur College

Business Mathematics (4th Ed.). By C. C. Richtmeyer and J. W. Foust. McGraw-Hill, New York, 1958. xii+412 pp. $5.75.

This college textbook is designed for students who are preparing to teach commercial arithmetic in secondary schools or to enter commercial fields and for other students who wish to become acquainted with the applications of elementary mathematics in business. The first four chapters review operations with integers, common fractions, decimal fractions, approximate numbers and

percentage. Other chapters pertain to simple interest and bank discount, graphs, measurement, common logarithms, arithmetic progressions and short-term buying, geometric progressions and compound interest, annuities, equations and statistics. This is an appropriate selection of topics and the sequence is excellent from the pedagogical point of view.

The simple, concise exposition includes many illustrations. Each chapter contains numerous verbal problems, a self-test and supplementary references. The problems are varied and interesting and supply additional information about current business practices. Tables are provided in the text. The format is excellent.

On the whole this is a text which the student can read with understanding. However, the instructor should note that in some instances the terminology is not precise; for example, the word "number" should be replaced by "integer" in some of the statements of section 11.

It is the opinion of this reviewer that chapters 1–3 are too elementary and that this treatment of numbers does not provide adequate preparation for the teaching of arithmetic in the secondary school. The chapters pertaining to graphs, installment buying, compound interest, annuities and statistics are excellent and may be recommended to all teachers of arithmetic.

Those who are familiar with earlier editions will find that in this revision the authors have extended and rewritten the problem material as well as the exposition of certain topics.

EDITH R. SCHNECKENBURGER
University of Buffalo

Intermediate Algebra for Colleges. By Gordon Fuller. Van Nostrand, New York, 1958. iv+258 pp. $3.90.

The claim is made that this book is an entirely new presentation of Intermediate Algebra. Such a strong statement is bound to be questioned, considering the number of books published on this subject. There are some outstanding features in this book. The author carefully builds on the foundation of the first chapter so that all the material follows readily, except for special topics. He has broken down most of the chapters into a large number of sections so that a minimum number of new ideas are encountered before a set of problems. In sections where rules are the end result, he gives examples that clearly bring out the rules. The justification of the laws of signed integers is well done. His explanation of stated problems is excellent. The chapter on logarithms is well done. The chapter on functions brings out implicit as well as explicit functions with a lucid explanation of the concept. Finally, he has a great number of problems from which to choose in nearly all the sections.

From a look at the chapters, one can see that the author has included the standard topics of Intermediate Algebra. The one possible exception to this is the lack of a discussion of rationals in either decimal or fractional form. One might criticize the book for the lack of this and other special topics. However,

there are always some topics omitted that some think should be included. On the whole, this is an above-average text.

LEONARD E. FULLER
Kansas State University

Zahlwort und Ziffer, Bd. I. By Karl Menninger. Vandenhoeck and Ruprecht, Göttingen, 1957. 221 pp. $4.00 (DM 16.80). Paperbound.

The author has rewritten and extended his original treatise on *A Cultural History of Number* into two volumes, of which this book is the first. Herein is treated the historical development of the number sequence (Zählreihe) and the number language. The second volume will treat the origin and development of number symbols, counting boards, and computation. The first part of this book presents a very scholarly and interesting picture of the cultural-historical growth of the sequence of number words or counting words. The words are traced back to their origin and the principles underlying their construction are supported with photographs, illustrations and primary historical sources.

The second half of the book deals with the origins of the spoken form of the number words in the many languages. Again there are ample illustrations and many tables of ordered number words to aid the reader. The author has taken special efforts to bring some light into the often cloudy region of etymology of number language. That the German word *zahl*, and the English words *tale* and *teller* all have the same root *tal*, meaning *to tell*, hence, *number*, is illustrative of much that is done with the other number words. There is detailed reference to the literature and an extensive bibliography of 112 references. The German is written in a simple and clear manner and can be read by a second year student of the language without too much difficulty. There is no comparable book in the English language.

HOWARD F. FEHR
Teacher's College, Columbia University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

Bradley University: Professor A. E. Gault has retired as Chairman of the Department of Mathematics and is succeeded by Professor M. G. Moore; Mr. Jack Barr, University of Iowa, and Captain G. C. Carlstedt, U. S. Coast Guard, Retired, have been appointed Instructors.

Vassar College: Associate Professor Winifred Asprey has been promoted to Professor;

Associate Professor Janet McDonald has been awarded a National Science Foundation Faculty Fellowship for 1959–1960 and will spend the year on leave at Indiana University; Miss Edith Moss, Wellesley College, has been appointed Instructor.

Mr. Maurice Anderson, University of Nebraska, has been appointed Instructor at the University of Wichita.

Mr. N. F. Beach, Eastman Kodak Company, Rochester, New York, has been promoted to Assistant Manager of film emulsion and plate manufacturing.

Dr. W. J. Berger, Raydist Navigation Corporation, Hampton, Virginia, has accepted a position as Research Scientist with Lockheed Missiles & Space Division, Palo Alto, California.

Assistant Professor R. C. Boles, North Carolina State College, has been appointed Professor of Mathematics and Chairman of the Mathematics Department at Tennessee Polytechnic Institute.

Mr. H. H. Brown, Lockheed Aircraft, Palo Alto, California, has accepted the position of Senior Mathematician with the Corporation for Economic and Industrial Research, Arlington, Virginia.

Assistant Professor P. L. Butzer, on leave from McGill University, is a Visiting Professor at the Technical University, Aachen, Germany.

Mr. F. P. Callahan, Jr., Philco Corporation, Philadelphia, Pennsylvania, has accepted a position as Mathematician with General Atronics, Bala-Cynwyd, Pennsylvania.

Dr. E. H. Connell, Lockheed Aircraft Corporation, Palo Alto, California, has been appointed Assistant Professor at the University of Miami.

Mr. E. L. Ellis, U. S. Army Ballistics Research Laboratory, Aberdeen, Maryland, has accepted the position of Mathematician-Programmer with Bendix Systems Division, Ann Arbor, Michigan.

Assistant Professor R. I. Fields, University of Louisville, has been promoted to Associate Professor and has been appointed Co-director of the University's Computing Laboratory.

Mr. A. L. Gilmore, Jr., Eglin Air Force Base, Florida, has accepted the position of Mathematician-Chief Programmer at the Electronic Computer Branch, Waterways Experiment Station, Vicksburg, Mississippi.

Dr. Arthur Grad, Office of Naval Research, has been appointed Program Director for Mathematical Sciences, Division of Mathematical, Physical, and Engineering Sciences of the National Science Foundation.

Assistant Professor R. T. Gregory, University of California, Goleta, has been appointed Associate Professor at the University of Texas.

Dr. J. J. Harton, Jr., Motorola, Inc., Riverside, California, has accepted the position of Senior Dynamics Engineer with Convair, San Diego, California.

Mr. D. L. Klippenstein, Korea Mennonite Central Committee, Taegu, Korea, has accepted the position of Programmer in the Computer Section of the Babcock and Wilcox Company, Barberton, Ohio.

Dr. U. R. Kodres, Iowa State College, has accepted a position as Associate Engineer for International Business Machines Corporation, Poughkeepsie, New York.

Professor Djuro Kurepa, head of the Institute of Mathematics, University of Zagreb, Jugoslavia, will lecture in the 1959 Summer Session at Teachers College, Columbia University.

Mr. S. L. Lida, formerly Applied Science Representative for the International Business Machines Corporation, New York, New York, has been appointed Manager, Communications Systems Marketing for the I.B.M. Corporation, White Plains, New York.

Visiting Professor G. G. Lorentz, University of Tubingen, Germany, has been appointed Professor at Syracuse University.

Mr. B. J. McDonald, Research Division of the U. S. Navy Mine Defense Laboratory in Panama City, Florida, is now associated with the Mathematical Analysis Section of Eglin Air Force Base, Florida.

Dr. Katsumi Okashimo, University of Toronto, is now a Defence Research Board Representative at the Computation Centre, McLennan Laboratory, University of Toronto.

Dr. Gideon Peyser, Newark College of Engineering, has been appointed Associate Professor at Pratt Institute.

Mr. E. H. Primoff, Institute of Mathematical Sciences, New York University, has accepted a position as Programmer with G. C. Dewey, New York City.

Mr. D. G. Russell, Shell Oil Company, Midland, Texas, has accepted the position of Reservoir Engineer in the Production Research Division of Shell Development Company, Bellaire, Texas.

Dr. D. R. Shreve, Carter Oil Company, Tulsa, Oklahoma, has been appointed Associate Professor at Oklahoma State University.

Mr. Irwin Stoner, Raytheon Manufacturing Company, Bedford, Massachusetts, has accepted the position of Senior Engineer with the Computation Laboratory, Arma Division, American Bosch Arma Corporation, Garden City, New York.

Dr. W. E. Wilson, Pratt Institute, has been appointed Chairman of Engineering, Harvey Mudd College of Science and Engineering.

Associate Professor O. M. Rasmussen, University of Denver, died on June 20, 1958. He was a member of the Association for eleven years.

SUMMER SESSIONS

Michigan State University, June 24–September 4: Professor Hocking, sets and abstract spaces, topology III; Professors Larcher and Doyle, theory of equations; Professor Parkus, thermal stresses; Professor Stelson, differential equations, advanced mathematics for engineers; Professor Wasserman, vector mechanics, numerical analysis; Professor Weeg, matrices and groups. June 24–July 30: Professor Hall, potential theory; Professor Norman, partial differential equations; Professor Stewart, theory of numbers, higher algebra III. July 31–September 4: Professor Doyle, complex variables II.

University of Buffalo, June 29–August 8: Professor Olson, history of mathematics; Professor Schneckenburger, infinite series, modern secondary mathematics.

University of California, Los Angeles, June 18–July 28: Visiting Professor Mackey, mathematical foundations of quantum mechanics; Professor Horn, introduction to advanced analysis; Visiting Professors Johnson and Lay, fundamental mathematical concepts (this last course is designed for secondary school teachers).

University of Illinois, June 15–August 8: Professor Suzuki, linear algebra; Professor Day, elementary geometry from a modern viewpoint; Professor Ketchum, functions of real variables. In addition, the following courses will be given: fundamental concepts, advanced algebra, introduction to higher algebra, complex variables and applications, introduction to higher analysis, advanced statistics, introduction to numerical analysis.

University of Texas, June 16–August 20: Professor Cooper, theory of functions of a complex variable; Professor Ettlinger, foundations of differential equations with applications, topics in teaching problems in mathematics; Professor Greenwood, boundary value problems; Profesor Guy, Fourier and Laplace transforms, topics in modern mathematics; Mr. Hurt, vector and tensor analysis; Professor Lane, mathematical statistics; Professor Lubben, introduction to modern projective geometry, topics in modern algebra; Professor Moore, theory of sets; Professor Wall, infinite processes, functions of a complex variable; Mrs. Barnes, topics in elementary mathematics; Staff, conference course, thesis, dissertation.

ANNUAL MEETING OF ASEE

The Mathematics Division of the American Society for Engineering Education will meet on Tuesday, June 16, 1959 at the University of Pittsburgh in conjunction with the 67th annual meeting of the ASEE. The following program will be presented.

Luncheon and Business Meeting at 12 noon: *A mathematician in an engineering school*, W. G. Warnock, Rensselaer Polytechnic Institute.

Joint Meeting with Physics Division at 2:00 P.M.: *An experimental program in teaching physics to engineers*, R. Resnick, Rensselaer Polytechnic Institute; *The teaching of modern mathematics to engineers*, J. P. Russell, Polytechnic Institute of Brooklyn; *Developments in the teaching of physics to engineers at Yale*, W. W. Watson, Yale University; *Coordinating the contents of mathematics and physics courses*, P. H. Cook, Pratt Institute.

Joint Dinner with Industrial Engineering Division at 6:30 P.M.: *Systems simulation*, Richard Bellman, Rand Corporation.

INTERNATIONAL CONGRESS OF MATHEMATICIANS 1962

The Secretary of the International Mathematical Union has announced that:

The Committee authorized by the final plenary session of the Congress in Edinburgh to determine the location of the International Congress 1962 has accepted an invitation from the Swedish National Committee for Mathematics and the Swedish Mathematical Society in the following terms:

"To mathematicians of all countries.

The Swedish National Committee for Mathematics and the Swedish Mathematical Society have the honour of inviting you to the next International Congress of Mathematicians, to be held in Stockholm during the Summer of 1962.

We will do our best to make the Congress scientifically successful and enjoyable, hoping that it will stimulate the interaction between mathematicians in different fields and countries. Ake Pleijel, Chairman of the Swedish National Committee for Mathematics, and Goran Borg, Chairman of the Swedish Mathematical Society."

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary Treasurer, announces that the following 270 persons have been elected to membership by the Board of Governors on applications duly certified.

MOSTAFA A. ABDELKADER, B.S. (Cairo) 9, Sh. Hamadan, Giza, Egypt.

ALLEN T. ADAMSON, M.A. (Whitman) Head of Dept., Cashmere High School, Washington.

ORVILLE W. ADDINGTON, M.A. (Geo. Washington) Asso. Professor, Virginia Polytechnic Institute.

CHARLES W. ALBERT, Student, University of British Columbia.

JACK ALVAREZ, B.E.E. (Georgia Tech.) Engineer, Boeing Airplane Co.

JEAN A. ANCTIL, M.A. (Laval) Head of Dept., College Militaire Royal de St. Jean.

ALLEN E. ANDERSEN, Ph.D. (Harvard) Head of Dept., University of Massachusetts.

JOHN D. ARRISON, M.A. (Michigan S.U.) Asst. Professor, Iowa Wesleyan College.

VERE AULIE, M.A. Teacher, South Tahoe High School, Altahoe, California.

GEORGE H. AUSTIN, M.S. (Wisconsin) Analyst, Investors Diversified Services.

RAYMOND G. AYOUB, Ph.D. (Illinois) Asso. Professor, Pennsylvania State University.

B. LOU ANN BARNO, Student, Oklahoma State University.

STANLEY L. BASIN, Student, San Jose State College.

BRO. CHRISTOPHER BEAULIEU, F.S.C. B.A. (Catholic) Teacher, St. Augustine High School, Brooklyn, New York.

CAMERON C. BOGUE, M.A. (Michigan) Instr., California State Polytechnic College.

JAMES C. BOLEN, M.S. (A. & M. Coll. of Texas) Instr., Agricultural and Mechanical College of Texas.

ROBERT W. BOLL, B.S. (St. Louis) Grad. Fellow, St. Louis University.

HAROLD E. BOWIE, M.A. (Maine) Professor and Head of Dept., American International College.

MARK C. BREITER, M.A. (Geo. Washington) Mathematician, Wright-Patterson Air Force Base.

KAREN D. BRENDER, Student, State University of Iowa.

HARRIET BROOKS, M.A. in Ed. (Clark) Chairman of Dept., Auburn High School, Massachusetts.

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REPORT OF THE TREASURER FOR THE YEAR 1958

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1958. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the Buffalo office of the Association.

There was a surplus of $5,701 in the Current Fund of the Association for 1958. The balances in the regular funds of the Association have increased during 1958 except for the Chace Fund. The cost of printing Slaught Papers and the third edition of Professional Opportunities in Mathematics has caused a small decrease in this fund.

Assets of the Association	January 1, 1958	December 31, 1958
M & T Trust Co., Buffalo	$ 25,703.17	$ 19,687.37
Savings Accounts	75,999.51	108,363.39
Securities	100,974.12	169,620.23
	$202,676.80	$297,670.99
Funds of the Association		
Current Fund	$ 325.80	$ 1,027.14
Carus Fund	20,235.25	26,706.76
Chace Fund	8,808.28	8,760.58
Houck Fund	11,309.92	13,832.45
Chauvenet Fund	1,328.43	1,620.95
Dunkel Fund	15,889.55	19,451.65
General Fund	38,414.63	51,428.83
	$ 96,311.86	$122,918.26
Visiting Lecturers Fund	$ 48,424.53	$ 38,285.38
Fund for Committee on Undergraduate Program	55,669.95	88,691.79
Fund for Committee on Films	85.66	—
Fund for Committee on High School Contests	2,184.80	427.34
Washington Conference Fund	—	2,337.42
Survey of Non-Teaching Mathematical Employment	—	1,218.43
Secondary School Lecturers Fund	—	16,054.92
Fund for Committee on Production of Films	—	27,827.45
	$202,676.80	$297,670.99

THE NOVEMBER MEETING OF THE INDIANA SECTION

The Fall meeting of the Indiana Section of the Mathematical Association of America was held at Marian College, Indianapolis, on November 6, 1958. The program was as follows:

Professor Judah Rosenblatt of Purdue University gave a one hour lecture on *Statistics and Aircraft Warning Systems.*

Professor Charles Brumfiel of Ball State Teachers College reported on the 13th annual national T.E.P.S. conference at Bowling Green State University.

Professor J. C. Polley, Wabash College, described the activities of the Indiana School and College Committee on Mathematics.

A panel composed of Professors Charles Brumfiel, Melvin Henriksen of Purdue University, Donald Lewis of the University of Notre Dame, and George Whaples of Indiana University, and moderated by Professor Merrill Shanks of Purdue University, discussed proposals of the M.A. A. Committee on the Undergraduate Program.

CHARLES BRUMFIEL, *Secretary*

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twenty-first annual meeting of the Northern California Section of the Mathematical Association of America was held at Stanford University, January 17, 1959. Professor B. J. Lockhart, Chairman of the Section, presided at the morning session and Professor G. C. Preston, Vice-Chairman of the Section, presided at the afternoon session. There were 134 persons in attendance, including 96 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor G. C. Preston, San Jose State College; Vice-Chairman, Professor S. P. Hughart, Sacramento State College; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College.

By invitation of the section, Professor J. L. Snell, Dartmouth College and Stanford University, delivered an address at the morning session entitled *Markov Chains and Their Applications*. An abstract of this address follows:

Recent applications of mathematics to the social sciences have given a renewed interest to the study of finite Markov chains. A procedure developed with J. G. Kemeny for systematically computing many of the basic descriptive quantities for a Markov chain was described. A discussion of some of the new applications of Markov chains was given.

At the end of the afternoon session a panel discussion was held by the newly-appointed Committee to Study the Activities of the Section consisting of Professor David Blakeslee, Chairman, Professors Henry Alder, Roy Dubisch, Harley Flanders, J. G. Herriot, Marjorie Hoffman, Brooks Lockhart, G. C. Preston, and Messrs. Kenneth Skeen and E. H. Swift. Some of the topics discussed were: The type of program most desirable for the regular meetings; what can we do to cooperate more fully with other organizations; the possibility of joint meetings with the California Mathematics Council; what would constitute the most workable executive committee; should we continue the lectureship program; should dues be collected and activities increased; should we issue a bulletin; how can we best extend our activities to include Hawaii; how can we cooperate with the Academy of Science in planning mathematics projects; and how can we work for the improvement of instruction in high schools.

Also, at this time, reports were given on the high school contest and lectureship program.

The following papers were presented:

1. *Acceptability in mathematics*, by Professor C. C. Torrance, U. S. Naval Postgraduate School, Monterey.

Is it *necessary* that a method be "logical" for it to be acceptable? Is it *sufficient* that a method produce "the right answer" for it to be acceptable? It is claimed here that the answer to both questions is *no*; consequent difficulties are discussed, and a method of resolution is indicated.

2. *Medial quasigroups*, by Professor D. A. Norton, University of California, Davis.

A short review of the concept of mediality, its significance and elementary properties.

3. *Roots and canonical forms of compound matrices*, by Dr. C. M. Ablow and Dr. J. L. Brenner, Stanford Research Institute.

Let A be a matrix in which the ith row R_i is obtained by applying the $(i-1)$th power of a permutation P to the sequence of elements $\{a_j\}_1^n$ of the first row. If P is the circular permutation of

$a_j \to a_{j-c}$, c an integer (subscripts being reduced modulo n), then A is called a c-circulant. The a_j can be numbers, or even $m \times m$ square matrices; in the latter case A is an $mn \times mn$ matrix. Properties of c-circulants are derived. In particular, some power of each latent root of any such matrix is shown to be obtainable as the root of a matrix of lower order. The results have applications in physics and chemistry.

4. *Strategies and learning models*, by Professor S. J. Bryant, Fresno State College, introduced by the Secretary.

In some learning experiments of the kind analyzed by Estes, it is shown that if the subject follows a certain type strategy his behavior will be similar to, and in some cases identical with, the behavior predicted by the Estes model.

5. *Some recent changes in the undergraduate mathematics program at the University of California, Berkeley*, by Professor Harley Flanders, University of California, Berkeley.

Several changes and experimental changes in the calculus and advanced calculus sequences were discussed, as was the new honors program in mathematics. Proposed additions to the upper division course list were mentioned.

ROY DUBISCH, *Secretary*

THE EARLE RAYMOND HEDRICK LECTURES

At the St. Louis meeting in December 1952 the Board of Governors of the Association voted to institute the custom of having a series of three expository lectures given at the summer meetings of the Association. It seemed appropriate to name these lectures after the late Professor Earle Raymond Hedrick, one of the founders and first president of the MAA. Each Hedrick lecturer is encouraged to publish his lecture as a Carus Monograph, a Slaught Paper, an article in this MONTHLY, or in other forms.

The following series of Hedrick Lectures have been delivered:

1952. Professor Tibor Rado, Ohio State University: "Derivatives and Jacobians."

1953. Professor P. R. Halmos, University of Chicago: "Axiomatic Set Theory."

1954. Professor L. H. Loomis, Howard University: "Convex Sets."

1955. Professor Mark Kac, Cornell University: "Familiar Things from an Unfamiliar Point of View."

1956. Professor J. C. Oxtoby, Bryn Mawr College: "Category and Measure."

1957. Professor Leo Zippin, Queens College: "Topological Transformation Groups."

1958. Dr. A. S. Householder, Oak Ridge National Laboratory: "Some Mathematical Problems Arising in Computations with Matrices."

1959. Professor William Feller, Princeton University (Invited).

THE CHAUVENET PRIZE

The Chauvenet Prize of the Association is awarded at three-year intervals for a noteworthy expository paper published during a preceding three-year period by a member of the Association. It is expected that the next prize will be awarded at the Annual Meeting to be held in January 1960 for a paper published during the period 1956–1958. The prize of $100 is awarded for a paper which will come within the range of profitable reading for members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars.

The Chauvenet Prize Fund of the MAA was established in 1925 by a contribution of $100 by Professor J. L. Coolidge, then President of the Association. There were subsequent gifts of $500 from Professor W. B. Ford and $100 from Professor Dunham Jackson. At various times the Association has also transferred money from its general funds to the Chauvenet Prize Fund.

The Chauvenet Prize has been awarded eleven times as follows:

1925 G. A. Bliss, "Algebraic Functions and Their Divisors," Annals of Mathematics.
1929 T. H. Hildebrandt, "The Borel Theorem and Its Generalizations," Bulletin of the American Mathematical Society.
1932 G. H. Hardy, "An Introduction to the Theory of Numbers," Bulletin of the American Mathematical Society.
1935 Dunham Jackson, "Series of Orthogonal Polynomials" and "Orthogonal Trigonometric Sums," Annals of Mathematics: "The Convergence of Fourier Series," this MONTHLY.
1938 G. T. Whyburn, "On the Structure of Continua," Bulletin of the American Mathematical Society.
1941 Saunders MacLane, "Modular Fields," and "Some Recent Advances in Algebra," both in this MONTHLY.
1944 R. H. Cameron, "Some Introductory Exercises in the Manipulation of Fourier Transforms," National Mathematics Magazine.
1947 P. R. Halmos, "The Foundations of Probability," this MONTHLY.
1950 Mark Kac, "Random Walk and the Theory of Brownian Motion," this MONTHLY.
1953 E. J. McShane, "Partial Orderings and Moore-Smith Limits," this MONTHLY.
1956 R. H. Bruck, "Recent Advances in the Foundations of Euclidean Plane Geometry," this MONTHLY.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to February 1, 1959)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary, a Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function

of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee, advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor, the Secretary and the Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of four members, including the Secretary and the Treasurer.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary, a Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary and the Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-

President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. In the absence of the President, the First Vice-President (or in his absence the Second Vice-President) shall have and exercise the powers of the President. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary shall have the usual duties pertaining to his office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President.

12. The Treasurer shall have the usual duties pertaining to his office including the collection of dues and the supervision and safekeeping of the funds of the Association.

Article IV—Meetings

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted. The by-laws of each Section when organized and any subsequent changes in these by-laws must be approved by the Board. The Board shall maintain general supervision over the activities of all Sections.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the admission fee.

2. The annual dues of each member shall be five dollars ($5), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of two dollars ($2).

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting. The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION AS OF FEBRUARY 1, 1959

(Except for the offices of President and Secretary-Treasurer, this list includes only the names of those who have held office since January 1, 1954. For information about preceding years, consult the American Mathematical Monthly for March 1957.)

PRESIDENT

E. R. HEDRICK	1916	ARNOLD DRESDEN	1933–1934
FLORIAN CAJORI	1917	D. R. CURTISS	1935–1936
E. V. HUNTINGTON	1918	A. J. KEMPNER	1937–1938
H. E. SLAUGHT	1919	W. B. CARVER	1939–1940
D. E. SMITH	1920	R. W. BRINK	1941–1942
G. A. MILLER	1921	W. D. CAIRNS	1943–1944
R. C. ARCHIBALD	1922	C. C. MACDUFFEE	1945–1946
R. D. CARMICHAEL	1923	L. R. FORD	1947–1948
H. L. RIETZ	1924	R. E. LANGER	1949–1950
J. L. COOLIDGE	1925	SAUNDERS MACLANE	1951–1952
DUNHAM JACKSON	1926	E. J. MCSHANE	1953–1954
W. B. FORD	1927–1928	W. L. DUREN, JR.	1955–1956
J. W. YOUNG	1929–1930	G. B. PRICE	1957–1958
E. T. BELL	1931–1932		

VICE-PRESIDENT

W. L. DUREN, JR.	1953–1954	R. V. CHURCHILL	1956–1957
H. S. M. COXETER	1954–1955	B. W. JONES	1957–1958
G. B. PRICE	1955–1956		

SECRETARY-TREASURER

W. D. CAIRNS	1916–1942	W. B. CARVER	1943–1947

ASSOCIATE SECRETARY

EDITH R. SCHNECKENBURGER 1948–1957

EDITOR

C. B. ALLENDOERFER 1952–1956

GOVERNOR (arranged alphabetically)

I. A. BARNETT	1952–1955	F. F. HELTON	1955–1958
C. F. BARR	1951–1954	M. S. HENDRICKSON	1955–1958
M. A. BASOCO	1954–1957	D. L. HOLL	1953–1954
H. W. BRINKMANN	1955–1957	A. S. HOUSEHOLDER	1956–1958
J. C. BRIXEY	1951–1954	AUGHTUM S. HOWARD	1951–1954
B. H. BROWN	1952–1955	R. C. HUFFER	1954–1957
S. S. CAIRNS	1953–1955	RALPH HULL	1954–1956
T. F. COPE	1951–1954	C. A. HUTCHINSON	1954–1957
H. H. DOWNING	1954–1957	S. B. JACKSON	1953–1956
J. M. EARL	1951–1954	R. D. JAMES	1952–1955
P. D. EDWARDS	1954–1957	L. W. JOHNSON	1954–1957
R. M. FOSTER	1954–1957	P. S. JONES	1953–1956
PHILIP FRANKLIN	1954–1956	F. W. KOKOMOOR	1955–1958
R. F. GRAESSER	1952–1955	D. H. LEHMER	1952–1954
E. R. HEINEMAN	1953–1956	F. A. LEWIS	1952–1955

L. L. Lowenstein	1955–1957	G. B. Price	1952–1955
K. O. May	1953–1956	J. F. Randolph	1955–1958
A. E. Meder, Jr.	1957–1958	C. B. Read	1955–1958
R. J. Michel	1952–1955	F. A. Rickey	1953–1956
E. B. Miller	1953–1956	E. B. Roessler	1951–1954
W. E. Milne	1952–1954	M. F. Smiley	1956–1958
C. W. Munshower	1952–1955	Ernst Snapper	1957–1958
W. H. Myers	1954–1957	C. E. Springer	1954–1956
Ivan Niven	1955–1958	F. H. Steen	1951–1954
C. O. Oakley	1953–1956	G. B. Thomas, Jr.	1955–1958
E. N. Oberg	1954–1956	C. W. Trigg	1953–1956
Morris Ostrofsky	1954–1957	A. W. Tucker	1953–1955
H. P. Pettit	1951–1954	M. A. Zorn	1955–1957
J. C. Polley	1951–1954		

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

Allegheny Mountain, University of Pittsburgh, May 2, 1959

Illinois, Millikin University, Decatur, May 8–9, 1959

Indiana, Valparaiso University, May 2, 1959

Iowa

Kansas

Kentucky

Louisiana-Mississippi

Maryland-District of Columbia-Virginia, Goucher College, Towson, Maryland, May 2, 1959

Metropolitan New York

Michigan

Minnesota

Missouri

Nebraska

New Jersey, Princeton University, November 7, 1959

Northeastern

Northern California, University of California, Berkeley, January 16, 1960

Ohio, Miami University, Oxford, May 9, 1959

Oklahoma

Pacific Northwest, University of Oregon, Eugene, June 19, 1959

Philadelphia, University of Delaware, Newark, November 28, 1959

Rocky Mountain, Utah State University, Logan, May 8–9, 1959

Southeastern

Southern California

Southwestern

Texas

Upper New York State, Hartwick College, Oneonta, May 9, 1959

Wisconsin, Wisconsin State College, Platteville, May 2, 1959

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CARRY OUT *engineering analysis of physical systems in electro-mechanical areas, deriving equations associated with systems study, developing generalized digital programs for parametric study. (PhD, MS)*

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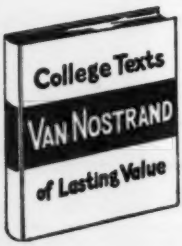

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